INTERNAL POLAR CONTINUUM THEORIES FOR SOLID AND FLUENT CONTINUA

A Dissertation

by

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ABSTRACT

Classical continuum theories are useful in the study of a variety of problems of engineering and applied sciences. However, the emergence of new materials has provided the need for refined theories that account for certain features that are not accounted for in the classical continuum theories. Polar decomposition of the deformation gradient tensor into pure stretch and pure rotation tensors shows that the rotation tensor will in general vary from point to point. Similarly, polar decomposition of the velocity gradient tensor shows that the rate of rotation tensor will vary from point to point. It can also be shown that the strain and strain rate tensors used in classical theories of continuum mechanics do not depend on the rotation tensor or its gradients and therefore neglect the effect of changing rotations and rates of rotations between neighboring material points in Lagrangian description, and between neighboring locations in Eulerian description. Varying rotations and rates of rotations between neighboring material points will, if resisted by the continua, result in internal moments which are conjugate to these rotations and rates of rotations. These internal moments along with the conjugate rotations and rates of rotations will result in energy storage and dissipation, in addition to the energy storage and dissipation resulting from stress and its conjugate strain and strain rate. Based on this observation, it is necessary to modify the existing conservation and balance laws to include internal moments, which results in a more complete thermodynamic framework for solid and fluent continua.

In this work, new conservation and balance laws are derived for solid and fluent continua that include internal moments which result from varying rotations and rotation rates. Also, constitutive theories are derived for the stress tensor, moment tensor, and heat vector, resulting in a complete mathematical model internal polar thermoelastic solids and internal polar thermoviscous fluids. This derivation does not rely on the introduction of external micro-rotations or stress couples as is done in the so called micro-polar or couple-stress theories. The theories presented here are therefore referred to as “internal polar continuum theories”, as they are derived using only internal measures of deformation and do not require introduction of external degrees of freedom. We also present a framework for obtaining approximate solutions to the mathematical models resulting from the new continuum theories. Numeric results are presented to show the affect of the internal polar theories presented here.

DEDICATION

To my Father, Mother, Brother, and Sister
with love.

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Portions of the literature review and the development of conservation and balance laws for polar solid continua were completed in conjunction with Dr. Daniel Nunez, post-doctoral researcher at the University of Kansas.

All other work conducted for the dissertation was completed by the student independently.

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NOMENCLATURE

x_i, $\boldsymbol{x}$, $\{x\}$	Coordinates in reference configuration
$\bar{x}_i$, $\bar{\boldsymbol{x}}$, $\{\bar{x}\}$	Coordinates in deformed (current) configuration
t	Time
$Q = Q(x_i, t)$	Lagrangian description of a quantity
$\bar{Q} = \bar{Q}(\bar{x}_i, t)$	Eulerian description of a quantity
$\dot{Q} = \frac{DQ}{Dt}$	Material derivative
${}^tQ = \frac{\partial Q}{\partial t}$	Ordinary time derivative
u_i, $\boldsymbol{u}$, $\{u\}$	Displacement
v_i, $\boldsymbol{v}$, $\{v\}$	Velocity
V	Volume
∂V	Closed surface of volume V
ρ_0	Density in reference configuration
ρ, $\bar{\rho}$	Density in current configuration
θ, $\bar{\theta}$	Temperature
$\bar{\boldsymbol{P}}$	Surface force per unit area
$\bar{\boldsymbol{F}}^b$	Body force per unit area
$\bar{\boldsymbol{M}}$	Surface moment per unit area
$\bar{\boldsymbol{n}}$	Unit outward normal vector
$\boldsymbol{e}_i$	Orthonormal basis vectors
$\tilde{\boldsymbol{g}}_i$	Covariant basis vectors
$\tilde{\boldsymbol{g}}^i$	Contravariant basis vectors
δ_{ij}	Kronecker delta symbol
ε_{ijk}	Permutation symbol
$\boldsymbol{a} \cdot \boldsymbol{b}$	Dot product
$\boldsymbol{A} : \boldsymbol{B}$	Double dot product
$\boldsymbol{a} \times \boldsymbol{b}$	Cross product
$\boldsymbol{a} \otimes \boldsymbol{b}$	Tensor (dyadic) product
$\boldsymbol{\nabla} \boldsymbol{a}$	Gradient operator
$\boldsymbol{\nabla} \cdot \boldsymbol{a}$	Divergence operator
$\boldsymbol{\nabla} \times \boldsymbol{a}$	Gradient operator
$[J]$, J_{ij}	Jacobian of deformation
$[J]$, J_{ij}	Jacobian of deformation
$[{}^dJ]$, ${}^dJ_{ij}$	Displacement gradient tensor
$[{}^d_sJ]$, ${}^d_sJ_{ij}$	Symmetric part of displacement gradient tensor
$[{}^d_aJ]$, ${}^d_aJ_{ij}$	Antisymmetric part of displacement gradient tensor
$[\varepsilon]$, $\boldsymbol{\varepsilon}$, ε_{ij}	Green's strain
$\boldsymbol{\Theta}$, Θ_i	Rotation tensor

$[{}^{\Theta}J]$, ${}^{\Theta}J_{ij}$, ${}^{\Theta}J_{ij}$	Rotation gradient tensor
$[\bar{L}]$, $\bar{\boldsymbol{L}}$, $\bar{L}_{ij}$	Velocity gradient tensor
$[\bar{D}]$, $\bar{\boldsymbol{D}}$, $\bar{D}_{ij}$	Symmetric part of velocity gradient tensor
$[\bar{W}]$, $\bar{\boldsymbol{W}}$, $\bar{W}_{ij}$	Antisymmetric part of velocity gradient tensor
$[\bar{L}]$, $\bar{\boldsymbol{L}}$, $\bar{L}_{ij}$	Velocity gradient tensor
$[\bar{D}]$, $\bar{\boldsymbol{D}}$, $\bar{D}_{ij}$	Symmetric part of velocity gradient tensor
$[\bar{W}]$, $\bar{\boldsymbol{W}}$, $\bar{W}_{ij}$	Antisymmetric part of velocity gradient tensor
$[{}^{\Theta}L]$	Rotation rate gradient tensor
$[{}^{\Theta}D]$	Symmetric rotation rate gradient tensor
$[{}^{\Theta}W]$	Antisymmetric rotation rate gradient tensor

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1. INTRODUCTION

1.1 Introduction and motivation

In Lagrangian description of deforming matter, the Jacobian of deformation is a fundamental quantity of the measure of deformation of solid continua. In general, the Jacobian of deformation varies between material points, i.e. it varies between a material point and its neighbors. Polar decomposition of the Jacobian of deformation at material points into stretch (left of right) and pure rotation shows that if the Jacobian of deformation varies between a material point and its neighbors so do the rotations. We could also consider the decomposition of the displacement gradient tensor into symmetric and skew symmetric tensors. The skew symmetric part of the displacement gradient tensor is a measure of pure rotations while the symmetric tensor is a measure of strains. Strain measures used in classical continuum mechanics (such as Green's strain) are purely a function of the stretch tensor or alternatively symmetric part of the displacement gradient tensor. In these measures, the rotation tensor plays no role. In classical, non-polar continuum theories, only conjugate stress and strain tensors contribute to the stored energy in the deforming solid continua. Likewise, the dissipation mechanism is purely due to the stress tensor and rates of the conjugate strain tensor. In such theories, the influence of rotations and the influence of the rates of rotations on the mechanism of energy storage and dissipation is not considered. In the current work, we consider solid continua in which the rotations and the rates of rotations that exist between neighboring material points are resisted by the constitution of the matter, hence result in energy storage and energy dissipation. Thus, the continuum theory presented here for solid continua in Lagrangian description incorporates new physics associated with varying internal rotations and their conjugate moments. This physics is completely absent in the currently used continuum theories for isotropic, homogeneous solid continua. This theory is a polar continuum theory that incorporates internal varying rotations and conjugate moments in the derivation of conservation and balance laws.

Similarly in deforming fluent continua, velocities and velocity gradients are fundamental quantities of the measure of deformation of the matter. In general, velocity gradients may vary between different locations i.e. they may vary between a location and its neighboring locations. Polar decomposition of the velocity gradient tensor at a location into rates of stretches (left or right) and rates of rotations shows that if the velocity gradient tensor varies between a location and the neighboring locations so does the rate of rotation tensor. We could also consider the decomposition of the velocity gradient tensor into symmetric and skew symmetric tensors. The skew symmetric tensor is a measure of pure rates of rotations while the symmetric tensor is a measure of strain rates. The measures of the internal rates of rotations due to deformation in these two approaches describe the same physics but in different forms. Polar decomposition gives the rates of rotation matrix and not the rates of rotation angles, whereas the skew symmetric part of the velocity gradient tensor yields rates of rotation angles that are explicitly defined in terms of velocity gradients. Strain rate measures are purely a function of stretch rates or alternatively symmetric part of the velocity gradient tensor. In these measures, the rate of rotation tensor plays no role. If these varying internal rates of rotations between neighboring locations in the deforming fluent continua are resisted by the continua then there must exist internal moments corresponding to these. The internal rates of rotations and the corresponding moments can result in rate of energy storage or rate of dissipation. This physics exists in all deforming fluent continua, but its degree may vary depending upon the constitution of the matter and the type of the deformation field. This physics is not considered in the derivation of conservation and balance laws that constitute the thermodynamic framework we are currently using for fluent continua. The answer to the question of what we should call the resulting continuum theory that incorporates the physics associated with internal rates of rotations and the corresponding moments is inherent in the description of the physics that the derivation of the theory incorporates. Since the theory accounts for internal rotation rates and associated moments, it is undoubtedly 'a polar continuum theory': (i) that only accounts for internal physics of rates of rotations resulting from the velocity gradient tensor and the conjugate moments (ii) that does not require rotations as additional external degrees of freedom as this theory is only intended to accommodate physics associated with internally varying rates of rotations that arise due to the varying velocity gradient tensor between points. Thus, henceforth we shall refer to the continuum theories presented here as 'internal polar continuum theories' implying that there may be others that account for different physics of rates of rotation and moments than considered here. In non-polar continuum theories used for fluent continua, stress and strain rates alone contribute to the dissipation i.e. entropy production due to mechanical work. In such theories the influence of varying internal rates of rotations is completely neglected, hence on the dissipation mechanism as well.

The theories presented here are continuum theories for solid and fluent polar continua and should not be confused with micropolar continuum theories [1–11] that are designed to accommodate effects at scales smaller than the continuum scale. Micropolar continuum theories require definitions of additional strain measures [6] related to micromechanics. Similarly, stress couple theories require the introduction of a direction kernel related to the non-local effects. The polar continuum theory presented here uses standard measures of strains as used currently in non-polar continuum theories. In the polar continuum theories presented here, the motivation is to account for the influence of varying rotations at neighboring material points that arises during evolution as these may result in additional energy storage in some solid continua, along with the influence of varying rates of rotations at spatial locations which may result in additional dissipation in some fluent continua. Polar decomposition of the Jacobian of deformation at neighboring material points clearly substantiates this. An important point to note is that the theory considered here can only account for local rotation effects due to deformation at material points, hence the theory presented here is intrinsically a local polar continuum theory, thus cannot account for nonlocal effects. While the necessity for these theories is motivated primarily by the fact that classical continuum theories do not incorporate the effects of varying rotations and rates of rotations, there are many applications that have shown the effect of micro-polar and stress couple physics. These include bone bending [12], cellular and porous solids [13–15], flows of binary fluids [16], and fluid suspensions [17]. The theories presented here could be applied to materials which exhibit similar physics, but have no concept of a length scale or microstructure.

1.2 Literature review

In the following we present a literature review on micropolar theories, nonlocal theories and stress couple theories. Even though some of these works may appear to have no direct connection with the work presented here, many of the concepts and derivation details in the cited references are quite helpful in following the details presented in this dissertation. The concept of couple stresses was introduced by Voigt in 1881 by assuming a couple or moment per unit area on the oblique plane of the deformed tetrahedron in addition to the stress or force per unit area. A more complete treatment of this theory and its relation to rotational degrees of freedom in a continuum was given by the Cosserat brothers [18], thus establishing the field of polar continuum mechanics. At this point, there is no direct relationship between rotations and displacements, in either a micro- or macro-sense. A comprehensive treatment of micromorphic continuum theories, of which micropolar theories are a subset, can be found in the works by Eringen [1, 3–9, 19–29]. Micromorphic continuum theories start with the assumption that displacement is the sum of a macro displacement vector and a micro displacement or director vector. The gradients of the macro-deformation vector are used to form the traditional strain measures, while the gradients of the micro-displacement vector form the so called micro-stretch and micro-rotation tensors. Balance laws for micromorphic materials are presented in reference [11]. The micropolar theories consider micro deformation due to micro constituents in the continuum. In references [30–32] by Reddy et al. and reference [33] by Zang et al. nonlocal theories are presented for bending, buckling and vibration of beams, beams with nanocarbon tubes and bending of plates. The nonlocal effects are incorporated due to the work presented by Eringen [6] in which definition of a nonlocal stress tensor is introduced through an integral relationship using the product of macroscopic stress tensor and a distance kernel representing the nonlocal effects. The polar continuum theories for solid and fluent continua presented here are strictly local and non-micropolar. Since the introduction of this concept many published works have appeared. We cite some recent works, most of which are related to micropolar stress couple theories. The concept of couple stresses is presented by Koiter [10]. Authors in reference [12] report experimental study of micropolar and couple stress elasticity of compact bones in bending. Conservation integrals in couple stress elasticity are reported in reference [34]. A microstructure-dependent Timoshenko beam model based on modified couple stress theories is reported by Ma et al. [35]. Further account of couple stress theories in conjunction with beams can be found in references [36–38]. Treatment of rotation gradient dependent strain energy and its specialization to Von Kármán plates and beams can be found in reference [39]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [13–15]. We remark that in references [12–15, 35–40], Lagrangian description is used for solid matter, however the mathematical descriptions are purely derived using strain energy density functional and principle of virtual work. This approach works well for elastic solids in which mechanical deformation is reversible, however extension of these works to thermoviscoelastic solids with and without memory is not possible. In such materials the thermal field and mechanical deformation are coupled due to the fact that the rate of work results in rate of entropy production. In reference [41] Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. The kinematics of plates is described using the vector of translations and the vector of rotations as dependent variables. Equations of equilibrium are established in $\mathbb{R}^3$ and $\mathbb{R}^2$. The strain energy density function is used to present linear constitutive theory. The mathematical models of reference [42] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [43], authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e. temperature can be treated as a parameter. In reference [44], authors present a collection of papers related to the mechanics of continua dealing with micro-macro aspects of the physics (largely related to solid matter). In reference [16] a micro-polar theory is presented for binary media with applications to phase-transitional flow of fiber suspensions. Such flows take place during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary layer flow of a polar fluid is given in reference [45]. In specific the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [46]. References [47, 48] consider material symmetry groups for linear Cosserat continuum and non-linear polar elastic continuum. Grekova et al. [49–51] consider various aspects of wave processes in ferromagnetic medium and elastic medium with micro-rotations as well as some aspects of linear reduced Cosserat medium. In references [18, 52–69] various aspects of the kinematics of micropolar theories, stress couple theories, etc. are discussed and presented including some applications to plates and shells.

In a series of papers published by Reddy et al. [30–32] and Zang et al. [70] nonlocal continuum theories are presented for bending, buckling, and vibration of beams, beam with carbon nano-tubes and bending of plates. The nonlocal effects are believed to be incorporated due to the work proposed by Eringen [6] in which definition of nonlocal stress tensor is introduced through integral relationship using the product of classical macroscopic stress tensor and a distance kernel representing the nonlocal effects. The polar continuum theory presented in this thesis is strictly local and non-micropolar. The references [30–32, 70] are cited here due to the fact that in many works nonlocal effects and micropolar theories are presented together, hence it is perhaps beneficial to understand the basic mechanisms advocated to incorporate nonlocal effects. The concept of couple stress was introduced by Voigt in 1881 by assuming a couple or moment per unit are on the oblique surface of the deformed tetrahedron in addition to the stress or force per unit area. Since the introduction of this concept many published works have appeared. We cite some recent works here most of which are related to micropolar couple stress theories. Experimental study of micropolar and couple stress elasticity in compact bone bending is reported in reference [12]. In [69] Yang et al. present a modification to couple stress theory in which they establish symmetry of the couple stress tensor by introducing the balance of moments of couples. They derive the balance of moments of couples based on the observation that the stress couples are local to a material point and cannot be freely translated and rotated as in classical mechanics. This leads to the notion of a higher order moment, moment of moments of force, or moment of couples which requires an additional balance law to ensure equilibrium. Conservation integrals in couple stress elasticity are reported in reference [40]. A microstructure dependent Timoshenko beam model based on modified couple stress theory is reported by Ma et al. [35]. Further accounts of couple stress theory in conjunction with beams can be forced in references [36–38]. Treatment of rotation gradient dependent strain energy and specialization for a von Kármán plates and beams can be found in [39]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [13–15]. We remark that references [12–15, 30–32, 35–40, 70], but more specifically [12–15, 35–40] consider solid matter in Lagrangian description, hence much of those concepts and derivations can not directly be adopted for fluent media. For example in references [35–39] use of strain energy density function, principle of virtual work etc. that are limited to solid matter, hence cannot be adopted in the work in this thesis. Even for solid matter, the works in references [13–15, 35–39] can only be used for thermoelastic solids (with small deformation in most cases) in which the mechanical deformation is reversible. These concepts and derivations cannot be used for thermoviscoelastic solids with or without memory as in such cases the deformation process is not reversible. We also remark that rotation gradient theory of [39] and others cited here for solid matter are not applicable for fluent continua considered in this paper as the displacements of the material points are not available and the fluent continua require consideration of varying internal rotation rates due to varying velocity gradient tensor between neighboring locations without regards to displacements. In reference [41] Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. Kinematics of plates is described using the vector of translations and the vector of rotations as dependent variables. Equations of equilibrium are established in $\mathbb{R}^3$ and $\mathbb{R}^2$. Strain energy density function is used to present linear constitutive theory. The mathematical models of reference [42] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [43], authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e. temperature can be treated as a parameter. In reference [44], authors present a collection of papers related to the mechanics of continua dealing with micro-macro aspects of the physics (largely related to solid matter). In reference [16] a micro-polar theory is presented for binary media with applications to phase-transitional flow of fiber suspensions. Such flows take place during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary layer flow of a polar fluid is given in reference [45]. In specific the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [46]. References [47, 48] consider material symmetry groups for linear Cosserat continuum and non-linear polar elastic continuum. Grekova et al. [49–51] consider various aspects of wave processes in ferromagnetic medium and elastic medium with micro-rotations as well as some aspects of linear reduced Cosserat medium. In references [18, 52–69] various aspects of the kinematics of micropolar theories, stress couple theories, etc. are discussed and presented including some applications to plates and shells.

Based on the literature review we make some remarks. First, most literature is related to micropolar theories that require consideration of additional measures of strains related to micromechanics. Such theories necessitate (micro-)rotations or rates of rotations as additional degrees of freedom. Conjugate to the rotations or rates of rotations are of course moments. In case of so called stress couple theories the physics considered is not clear at the onset. It is only after the derivation of balance laws, specifically the so called “conservation of inertia”, that one gets some idea regarding what these theories can possibly do.

The work presented here is formulated based on observed physics, that in any deforming continua the polar decomposition of the velocity gradient tensor shows that the rates of rotations vary between neighboring locations. If the varying rotation rates and their gradients result in energy storage or dissipation, then its energy conjugate moment tensor must exist in the deforming matter. This necessitates the existence of moment (per unit area) on the oblique plane of the deformed tetrahedron. Thus, at the onset, we consider average force per unit area and velocities, and average moment per unit area and the rates of rotations on the oblique plane of the deformed tetrahedron. The work presented here follows strictly thermodynamic approach i.e. for polar solid and fluent continua we present derivations of: (i) conservation of mass and present reasons for not deriving conservation of inertia (ii) balance of linear momenta (iii) balance of angular momenta (iv) balance of moments of moments (or couples) (v) first law of thermodynamics (vi) second law of thermodynamics based on stress and strain rates, moment and rotation rates as energy conjugate pairs and (vii) constitutive theories for stress tensor, moment tensor, and heat vector. The mathematical description for polar continua derived here is applicable to compressible and incompressible thermoviscous polar fluids and to compressible and incompressible polar thermoelastic solids. It can be applied to thermoviscoelastic polar fluids and thermoviscoelastic polar solids with or without memory when augmented with the appropriate constitutive theories. We reiterate that the polar theories for solid and fluent continua presented here incorporates additional physics due to rotations and rates of rotations which is neglected in the currently used thermodynamic framework. Thus this polar theory presents a more complete form of thermodynamic framework for isotropic, homogeneous solid and fluent continua. The currently used thermodynamic framework is retained as a subset of the thermodynamic framework presented here.

2. INTERNAL POLAR CONTINUUM THEORY FOR FLUENT CONTINUA*

2.1 Mathematical description for fluent continua

For a deforming volume of matter, whether solid or fluid, material particles and their motion i.e. displacements are the most fundamental quantities that describe the physics of deformation. If x_i is the position of a material particle in the reference configuration then its coordinates $\bar{x}_i$ in the current configuration can be determined using $\bar{x}_i = x_i + u_i$ in which u_i are the displacements. Based on this we can derive conservation and balance laws using a deformed tetrahedron in the current configuration (Fig. 2.1 (b)) and its corresponding undeformed counterpart in the reference configuration (Fig. 2.1 (a)). If the resulting equations are expressed as functions of x_i and t, then we have a Lagrangian description of motion. On the other hand, if the resulting equations are a function of $\bar{x}_i$ and t then we have an Eulerian description of motion. Due to the fact that $\bar{x}_i = x_i + u_i$, the Lagrangian and Eulerian descriptions are identical mathematical representations of the same physics. Using $\bar{x}_i = x_i + u_i$ we can easily convert one type of description to another type without any loss of information. At this stage the Lagrangian and the Eulerian descriptions are equally suited for solid as well as fluent continua and have total transparency in deriving one from the other. If some special consideration of the physics in a continua requires some modification in either one of the two descriptions, then the transparency between the two will obviously be lost. We consider specific cases in the following. Refer to reference [71] (Chapters 6 and 7) for additional details.

In the case of solids the material points are identified (x_i) and their displacements are monitored (u_i) hence $\bar{x}_i = x_i + u_i$ holds at each material point. Thus the Lagrangian and Eulerian descriptions are equivalent, and either one can be used for the mathematical description of the physics. Due to complex motion of fluid particles, monitoring of their motion i.e. displacements is not feasible. Thus, in the case of fluent continua, the first adjustment required by physics of complex motion is not to monitor material point displacements (u_i). This of course suggests that we do not know the whereabouts of the material points during evolution. Deformed positions $\bar{x}_i$ of the material points in the current configuration are only due to displacements u_i which we do not have anymore. Since we cannot monitor displacements of the material particles in fluent continua, it is perhaps fitting in case of fluent continua not to label the material points. Thus, in the case of fluent continua we ignore material point displacements i.e. the motion of the material points during the evolution. The only other alternative left at this stage is that we consider fixed locations in the flow at which we monitor the state of the continua (temperature, velocity, etc.) during evolution. These fixed locations are occupied by different fluid particles during evolution. Thus, we could view these locations as current positions of different fluid particles for different values of time. As time elapses the fluid particles currently occupying these positions leave their positions which in turn are occupied by other fluid

*Portions of the derivation of the conservation and balance laws presented in this chapter appear in the article "A Polar Continuum Theory for Fluent Continua" by K.S. Surana, J.N. Reddy, and M. Powell *Int. J. of Engg. Research & Indu. Appls. (IJERIA)* Vol. 8, No. 2, pp. 107–146 (2015) ©Ascent Journals. Portions of the derivation of the constitutive theories appear in the article "Ordered Rate Constitutive Theories for Internal Polar Thermofluids" by K.S. Surana, M. Powell, and J.N. Reddy *Int. J. of Math. Sci. & Engg. Appls. (IJMSEA)* Vol. 9, No. 3 pp. 51–116 (2015) ©Ascent Journals

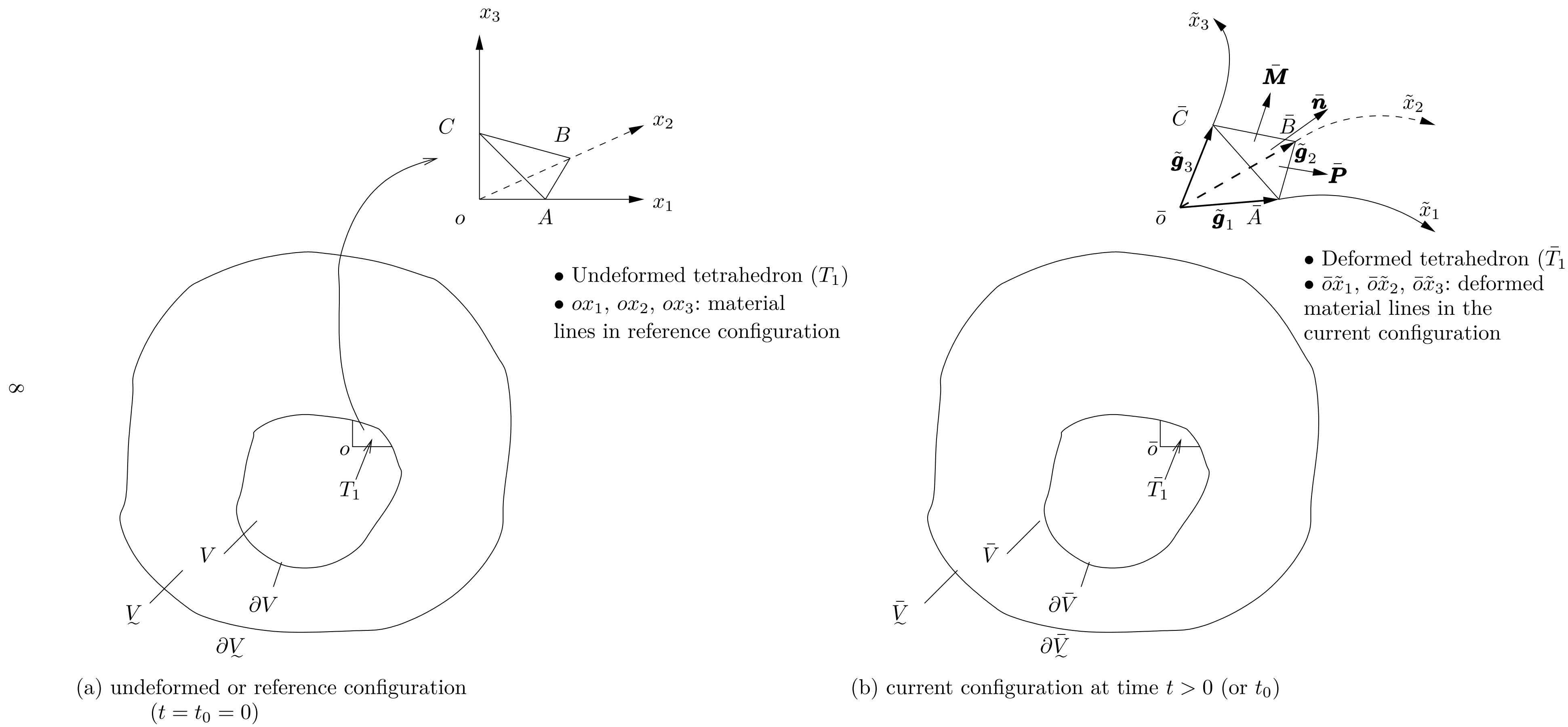


Figure 2.1: Reference and current configurations for a deforming volume of matter

particles. Here, there are two important things to note: (i) each fixed location is the current position of some fluid particle, hence it is appropriate to label these as $\bar{x}_i$, keeping in mind that there are no x_i as u_i are not monitored (ii) we do not know which fluid particles are at which locations. Monitoring the state of fluent continua (velocities, temperature, etc.) at each location describes the evolution of the deforming continuum.

We need to determine what mathematical model would be able to describe the physics that we have just discussed. Since the locations at which the evolution is monitored, though fixed, are current locations of different material particles at different values of time. This perhaps suggests that we can begin by choosing Eulerian description in which $\bar{x}_i$ are the fixed locations. In order for this mathematical model to be applicable for fluids, $\bar{u}_i$ and $\bar{u}_{i,j}$ must be eliminated. The resulting mathematical model does not contain u_i and x_i nor does it require their use. We must decide what to call this mathematical model, certainly not Eulerian, as a true Eulerian description requires x_i and u_i so that its counterpart Lagrangian description can be obtained transparently. In this model, u_i do not exist, hence neither do the strains. This is perfectly fine for fluids as in the case of fluid motion description displacements and strain measures play no role; instead velocities at $\bar{x}_i$ (fixed) and their gradients (strain rates) are fundamental measures of deformation. In summary we have: (i) Eulerian description in which $\bar{x}_i$ are fixed locations (ii) u_i (or $\bar{u}_i$) are not considered (iii) velocities $\bar{v}_i$ and its gradients $\frac{\partial \bar{v}_i}{\partial \bar{x}_j}$ are fundamental quantities in the kinematic description of motion using conservation and balance laws. This description is what is used currently in fluid mechanics. In the absence of $\bar{u}_i$ and x_i this description can not be a true Eulerian description. The origin of the derivation of this mathematical model is true Eulerian description with the restriction that we do not have $\bar{u}_i$ and x_i available to us. The derivation of the conservation and balance laws for polar fluent continua in this paper are presented utilizing this approach, i.e. configurations in figure 1 (a) and (b) are assumed to exist at the onset and during the derivation of conservation and balance laws, but at the end only the Eulerian description is retained with the restriction that $\bar{u}_i = 0$ and $\bar{x}_i$ in the model are the fixed locations at which the evolution is monitored. In simple terms we follow Eulerian description but ensure that x_i and $\bar{u}_i$ are not part of the final mathematical model. Thus, in all subsequent material in this paper use of 'Eulerian description' refers to what has been defined here as Eulerian description for fluent continua.

We use an over bar on quantities to express quantities in the current configuration in Eulerian description, that is, all quantities with over bars are functions of current coordinates $\bar{x}_i$ and time t. We denote $\bar{\rho}$ to be the density of the fluid in the current configuration and $\bar{\Phi}$, $\bar{\theta}$, and $\bar{\eta}$ denote the Helmholtz free-energy density, temperature, and entropy density, respectively. $\bar{\boldsymbol{\sigma}}^{(0)}$ is the Cauchy stress tensor in Eulerian description in contravariant basis. The superscript '0' is used to signify that it is rate of order zero and the parenthesis distinguish it from the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ used in Lagrangian description. Dot on any quantity refers to the material derivative.

If the existence of different rates of rotation at neighboring locations, as evident from the polar decomposition of the velocity gradient tensor, can result in additional mechanical energy dissipation, then there must also coexist energy conjugate moments in the deforming matter. Just like forces and velocities result in rate of work, moments and rates of rotation can also result in rate of work. Thus in the development of the polar continuum theory in Eulerian description for fluent media we consider existence of moments and rotation rates independent of forces and velocities. Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration (figure 2.1 (a)) with closed boundary $\partial\underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 shown in figure 2.1 (a) such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron $\bar{T}_1$ is placed back in the volume $\bar{V}$. Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short, and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions.

2.1.1 Velocity and rotation rate gradient tensors

Consider The velocity gradient tensor $\bar{\boldsymbol{L}}$ and its decomposition into symmetric and skew symmetric parts $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{W}}$

$$\bar{L}_{ij} = \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \quad \text{or} \quad [\bar{L}] = \left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right] = [\bar{D}] + [\bar{W}] \tag{2.1}$$

$$[\bar{D}] = \frac{1}{2}\left([\bar{L}] + [\bar{L}]^T\right); \quad [\bar{W}] = \frac{1}{2}\left([\bar{L}] - [\bar{L}]^T\right) \tag{2.2}$$

Let $\{{}^t\bar{\Theta}\} = [{}^t\bar{\Theta}_{x_1} \;\; {}^t\bar{\Theta}_{x_2} \;\; {}^t\bar{\Theta}_{x_3}]^T$ be the rates of rotation about ox_1, ox_2, and ox_3 axes of the x-frame, then we have

$$[\bar{W}] = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x_3} & -{}^t\bar{\Theta}_{x_2} \\ -{}^t\bar{\Theta}_{x_3} & 0 & {}^t\bar{\Theta}_{x_1} \\ {}^t\bar{\Theta}_{x_2} & -{}^t\bar{\Theta}_{x_1} & 0 \end{bmatrix} \tag{2.3}$$

in which

$$\begin{aligned} {}^t\bar{\Theta}_1 = {}^t\bar{\Theta}_{x_1} &= \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_2}\right) \\ {}^t\bar{\Theta}_2 = {}^t\bar{\Theta}_{x_2} &= \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_3}\right) \\ {}^t\bar{\Theta}_3 = {}^t\bar{\Theta}_{x_3} &= \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1}\right) \end{aligned} \tag{2.4}$$

We define gradients of ${}^t\bar{\boldsymbol{\Theta}}$ by

$$ {}^{\bar{\Theta}}\bar{L}_{ij} = \frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j}; \quad \left[{}^{\bar{\Theta}}\bar{L}\right] = \frac{\partial\{{}^t\bar{\Theta}\}}{\partial\{\bar{x}\}} = \left[{}^{\bar{\Theta}}\bar{D}\right] + \left[{}^{\bar{\Theta}}\bar{W}\right] \tag{2.5} $$

Symmetric and skew symmetric tensors $[{}^{\bar{\Theta}}\bar{D}]$ and $[{}^{\bar{\Theta}}\bar{W}]$ are defined by

$$ [{}^{\bar{\Theta}}\bar{D}] = \frac{1}{2}\left(\left[{}^{\bar{\Theta}}\bar{L}\right] + \left[{}^{\bar{\Theta}}\bar{L}\right]^T\right); \quad \left[{}^{\bar{\Theta}}\bar{W}\right] = \frac{1}{2}\left(\left[{}^{\bar{\Theta}}\bar{L}\right] - \left[{}^{\bar{\Theta}}\bar{L}\right]^T\right) \tag{2.6} $$

2.1.2 Polar decomposition of velocity gradient tensor and consideration of local rotation rates

Polar decomposition of the velocity gradient tensor decomposes deformation into the stretch rate tensor and rotation rate tensor. Whether we use left stretch rate tensor or right stretch rate tensor, the rotation rate tensor is unique. Thus, at each location with infinitesimal volume surrounding it, the velocity gradient tensor $[\bar{L}]$ can be decomposed into pure rates of rotation $[{}^t\bar{R}]$ and right or left stretch rate tensors $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$. $[{}^t\bar{R}]$ is orthogonal and $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$ are symmetric and positive definite. The rotation rate tensor can equivalently be obtained due to rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ at each location in the flow domain. Thus, at each location int the flow domain the rotation rate the values in the $[{}^t\bar{R}]$ matrix can be viewed as being due to ${}^t\bar{\boldsymbol{\Theta}}$. If varying rotation rates at varying locations in the flow domain are resisted by the constitution of the fluent continua then this must result in additional dissipation that requires existence of energy conjugate moments $\bar{\boldsymbol{M}}$ in the deforming matter. Thus, at the onset ${}^t\bar{\boldsymbol{\Theta}}$ and its conjugate $\bar{\boldsymbol{M}}$ are considered in the derivation of the polar continuum theory for the fluent continua. Details of polar decomposition of $[\bar{L}]$ and rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ are given in the following. Let

$$ [\bar{L}] = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{R}] \tag{2.7} $$

Let $({}^t\lambda_i, \{\phi\}_i);\ i = 1,2,3$ be the eigenvalues of $[\bar{L}]^T[\bar{L}]$ in which $\{\phi\}_i^T\{\phi\}_j = \delta_{ij}$, then

$$ \left[\bar{L}\right]^T \left[\bar{L}\right] = \left[\bar{\Phi}\right]\left[{}^t\bar{\lambda}\right]\left[\bar{\Phi}\right]^T = [{}^t\bar{S}_r]^2 \tag{2.8} $$

The columns of $[\bar{\Phi}]$ are eigenvectors $\{\phi\}_i$ and $[{}^t\bar{\lambda}]$ is a diagonal matrix of ${}^t\lambda_i,\ i = 1,2,3$. If we choose

$$ [{}^t\bar{S}_r] = \left[\bar{\Phi}\right]\left[\sqrt{{}^t\bar{\lambda}}\right]\left[\bar{\Phi}\right]^T \tag{2.9} $$

Then (2.8) holds, hence $[{}^t\bar{S}_r]$ can be defined using (2.9). $[{}^t\bar{R}]$ can now be determined using (2.7)

$$ [{}^t\bar{R}] = [\bar{L}][{}^t\bar{S}_r]^{-1} \tag{2.10} $$

Thus, we have established $[{}^t\bar{R}]$ and $[{}^t\bar{S}_r]$ in polar decomposition (2.7). Using

$$ \left[\bar{L}\right]\left[\bar{L}\right]^T = [{}^t\bar{S}_l]^2 \tag{2.11} $$

and following a similar procedure we can establish the following

$$[{}^t\bar{S}_l] = \left[\bar{\Phi}\right]\left[\sqrt{{}^t\bar{\lambda}}\right]\left[\bar{\Phi}\right]^T \tag{2.12}$$

$$[{}^t\bar{R}] = [{}^t\bar{S}_l]^{-1}[\bar{L}] \tag{2.13}$$

in which $({}^t\lambda_i, \{\phi\}_i)$; $i = 1,2,3$ are eigenpairs of $[\bar{L}][\bar{L}]^T$. $[{}^t\bar{R}]$ defined by (2.10) or (2.13) is unique. The rate of rotation matrix $[{}^t\bar{R}]$ can equivalently be obtained due to rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ at each location. Thus, at each location $[{}^t\bar{R}]$ can be viewed as being due to rates of rotations ${}^t\bar{\boldsymbol{\Theta}}$. Rate of energy dissipation due to ${}^t\bar{\boldsymbol{\Theta}}$ requires coexistence of moments $\bar{\boldsymbol{M}}$ (per unit area) on the oblique surface of the tetrahedron in the deforming matter. Thus we have

$$[\bar{L}] = \frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}} = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{R}] \tag{2.14}$$

where

$$[{}^t\bar{R}] = \left[{}^t\bar{R}({}^t\bar{\boldsymbol{\Theta}})\right] \tag{2.15}$$

Explicit forms of ${}^t\bar{\boldsymbol{\Theta}}$ i.e. ${}^t\bar{\Theta}_{x_1}$, ${}^t\bar{\Theta}_{x_2}$, and ${}^t\bar{\Theta}_{x_3}$ or ${}^t\bar{\Theta}_1$, ${}^t\bar{\Theta}_2$, and ${}^t\bar{\Theta}_3$ in terms of velocity gradients are given in section 2.1.1.

2.1.3 Rotation rate gradients and strain rate gradients

Even though the presence of varying rates of rotations between neighboring locations in the flow domain may influence the dissipation in some fluent continua, the precise manner in which this occurs is not yet established. All we know at this stage is that in fluent continua, in addition to forces and velocities, the rotation rates and moments can also be rate of work conjugate if the deforming fluent continua resists varying rotation rates between between the neighboring locations in the flow domain. Through the derivations of the balance laws presented in section 2.2 we establish that the symmetric part of the rotation rate gradient tensor is energy conjugate to the moment tensor. Thus, it is accurate to say that the polar part of the theory presented here is due to rates of rotation gradients. The purpose of the material in this section is to demonstrate that the polar continuum theory presented here is not the same as the strain rate gradient theory published or referenced in the literature.

In the case of solid matter, Shield [72] shows a relationship between the gradients of local rotations in terms of gradients of strain tensor and rotation tensor. Based on similar works, it is argued and mostly accepted that the continuum theories that incorporate rotation gradients are same as those derived using strain gradients. In the following we present a derivation for fluent continua to demonstrate that the theories based on rotation rate gradients are not the same as those that are derived using strain rate gradients. This is necessary to differentiate the work presented in this dissertation from the published works on strain rate gradient theories. For simplicity consider a two dimensional state of deformation in the x_1x_2-plane. The velocity gradient tensor $[\bar{L}]$ is given by

$$[\bar{L}] = \left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right] = [\bar{D}] + [\bar{W}] \tag{2.16}$$

where $[\bar{D}]$ and $[\bar{W}]$ are symmetric and skew symmetric tensors.

$$[\bar{W}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1}\right) \\ \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x_3} \\ -{}^t\bar{\Theta}_{x_3} & 0 \end{bmatrix} \tag{2.17}$$

in which

$${}^t\bar{\Theta}_{x_3} = \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1}\right) = {}^t\bar{\Theta}_3 \tag{2.18}$$

is the rate of rotation tensor about the x_3 axis. Gradients of ${}^t\bar{\Theta}_3$ with respect to $\bar{x}_1$ and $\bar{x}_2$ are

$$\begin{aligned} {}^t\bar{\Theta}_{3,1} &= \frac{1}{2}\left(\frac{\partial^2 \bar{v}_1}{\partial \bar{x}_1 \partial \bar{x}_2} - \frac{\partial^2 \bar{v}_2}{\partial \bar{x}_1^2}\right) \\ {}^t\bar{\Theta}_{3,2} &= \frac{1}{2}\left(\frac{\partial^2 \bar{v}_1}{\partial \bar{x}_2^2} - \frac{\partial^2 \bar{v}_2}{\partial \bar{x}_1 \partial \bar{x}_2}\right) \end{aligned} \tag{2.19}$$

The strain rates are defined by $[\bar{D}]$ (same in co- and contra-variant bases and Jaumann rates)

$$[\bar{D}] = \begin{bmatrix} \frac{\partial \bar{v}_1}{\partial \bar{x}_1} & \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} + \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) \\ \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} + \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) & \frac{\partial \bar{v}_2}{\partial \bar{x}_2} \end{bmatrix} = \begin{bmatrix} \dot{\bar{\varepsilon}}_{11} & \dot{\bar{\varepsilon}}_{12} \\ \dot{\bar{\varepsilon}}_{21} & \dot{\bar{\varepsilon}}_{22} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \tag{2.20}$$

in which $\gamma_{21} = \gamma_{12}$.
Substituting from (2.20) into (2.19) we can obtain

$$\begin{aligned} {}^t\bar{\Theta}_{3,1} &= \frac{\partial \gamma_{11}}{\partial \bar{x}_2} - \frac{\partial \gamma_{12}}{\partial \bar{x}_1} \\ {}^t\bar{\Theta}_{3,2} &= \frac{\partial \gamma_{12}}{\partial \bar{x}_2} - \frac{\partial \gamma_{22}}{\partial \bar{x}_1} \end{aligned} \tag{2.21}$$

In (2.21), the gradients ${}^t\bar{\Theta}_{3,1}$ and ${}^t\bar{\Theta}_{3,2}$ of rotation rate ${}^t\bar{\Theta}_3$ are completely expressed in terms of the gradients of γ_{11} and γ_{22} with respect to $\bar{x}_2$ and $\bar{x}_1$ and γ_{12} with respect to $\bar{x}_1$ as well as $\bar{x}_2$

Remarks

1. From (2.21) we note that gradients of ${}^t\bar{\Theta}_3$ are functions of $\frac{\partial \gamma_{11}}{\partial \bar{x}_2}$, $\frac{\partial \gamma_{22}}{\partial \bar{x}_1}$, $\frac{\partial \gamma_{12}}{\partial \bar{x}_1}$, and $\frac{\partial \gamma_{12}}{\partial \bar{x}_2}$ but are not functions of $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$. This is expected due to the fact that $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ are gradients of elongation rates per unit length in $\bar{x}_1$ and $\bar{x}_2$ directions, hence can not possibly contribute to the gradients of the rotation rates.

2. Consideration of ${}^t\bar{\Theta}_{3,1}$ and ${}^t\bar{\Theta}_{3,2}$ in polar theory is identically equivalent to replacing these by the right sides of the expressions in (2.21). As long as this condition is satisfied the polar theory based on the gradients of rotation rates is the same as the polar theory based on gradients of the strain rates. We keep in mind that $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ are not part of the expressions of the gradients of rotation rates in (2.21).

3. A polar theory based on strain rate gradients must consider $\gamma_{ij,k}$ i.e. gradients of all six strain

rates with respect to $\bar{x}_k$. Thus at the onset it is clear that the strain rate gradient theory for 2D cases will also consider $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ in the derivation in addition to the other strain rate gradients that appear in (2.21). If we consider three dimensional case (i.e. $\mathbb{R}^3$) then we would find that additionally $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ will appear in the strain rate gradient theory but will be absent in the definitions of the gradients of the rotation rates.

4. The rotation rate polar theory resulting due to consideration of local rotation rates is targeted towards specific physics of rotation rates resulting in additional dissipation in a deforming fluent continua. We have shown that a polar theory based on gradients of rates of rotations is not the same as the theories derived using gradients of strain rates. We remark that equation (2.21) representing gradients of rotation rates as a function of some (and not all) of the gradients of strain rates is a consequence of mathematical manipulation.

2.1.4 Covariant and Contravariant bases

The edges of the deformed tetrahedron $\bar{T}_1$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that are tangent to the deformed material lines at $\bar{o}$. The faces of the tetrahedron are formed by the covariant base vectors $\tilde{\boldsymbol{g}}_2,\tilde{\boldsymbol{g}}_3$; $\tilde{\boldsymbol{g}}_3,\tilde{\boldsymbol{g}}_1$; and $\tilde{\boldsymbol{g}}_1,\tilde{\boldsymbol{g}}_2$. Following [71, 73, 74] we can define

$$\tilde{\boldsymbol{g}}_i = \frac{\partial \bar{x}_k}{\partial x_i}\boldsymbol{e}_k \tag{2.22}$$

x_i and $\bar{x}_k$ being coordinates of a material point in the reference configuration and current configuration respectively. If $[J]$ is the Jacobian of deformation

$$[J] = \frac{\partial\{\bar{x}\}}{\partial\{x\}} \quad \text{or} \quad J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \tag{2.23}$$

then the columns of $[J]$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$. The contravariant basis are reciprocal to the covariant basis [71, 73, 74] and are defined by the base vectors $\tilde{\boldsymbol{g}}^i$

$$\tilde{\boldsymbol{g}}^j = \frac{\partial x_j}{\partial \bar{x}_l}\boldsymbol{e}_l \tag{2.24}$$

We note that

$$\tilde{\boldsymbol{g}}_i \cdot \tilde{\boldsymbol{g}}^j = \delta_{ij} \tag{2.25}$$

Alternatively to (2.24) we can also define $\tilde{\boldsymbol{g}}^i$ as

$$\tilde{\boldsymbol{g}}^1 = \frac{\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3}{\tilde{\boldsymbol{g}}_1 \cdot (\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3)}, \quad \tilde{\boldsymbol{g}}^2 = \frac{\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1}{\tilde{\boldsymbol{g}}_2 \cdot (\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1)}, \quad \tilde{\boldsymbol{g}}^3 = \frac{\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2}{\tilde{\boldsymbol{g}}_3 \cdot (\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2)} \tag{2.26}$$

The volume of the parallelepiped framed by $\tilde{\boldsymbol{g}}_i$ in the current configuration is given by (same as denominators in 2.26)

$$\bar{V} = \tilde{\boldsymbol{g}}_1 \cdot (\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3) = \tilde{\boldsymbol{g}}_2 \cdot (\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1) = \tilde{\boldsymbol{g}}_3 \cdot (\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2) \tag{2.27}$$

We note that $\tilde{\boldsymbol{g}}^i$ in (2.24) as well as $\tilde{\boldsymbol{g}}^j$ in (2.26) satisfy (2.25). Thus definitions of $\tilde{\boldsymbol{g}}^j$ in (2.24) and (2.26) are exactly the same, as both definitions with (2.22) satisfy (2.25). We note that $\tilde{\boldsymbol{g}}^1$, $\tilde{\boldsymbol{g}}^2$, $\tilde{\boldsymbol{g}}^3$ are normal to the faces of the deformed tetrahedron formed by $\tilde{\boldsymbol{g}}_2, \tilde{\boldsymbol{g}}_3$; $\tilde{\boldsymbol{g}}_3, \tilde{\boldsymbol{g}}_1$; $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$ covariant base vectors. Covariant and contravariant directions are important in defining and choosing the correct measures of strains, stresses, moment intensities, etc. Under the action of $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ on surface $\bar{A}\bar{B}\bar{C}$ and the stress and moment intensities on the faces of the tetrahedron formed by $\tilde{\boldsymbol{g}}_2$, $\tilde{\boldsymbol{g}}_3$; $\tilde{\boldsymbol{g}}_3, \tilde{\boldsymbol{g}}_1$; and $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$ base vectors, the tetrahedron $\bar{T}_1$ is in equilibrium.

2.1.5 Definition of stress measures

2.1.5.1 Contravariant Cauchy stress tensor

The definition of the stresses on the non-oblique faces of the tetrahedron in the contravariant directions is the most natural way to define stress. Let $\bar{\underset{\sim}{\boldsymbol{\sigma}}}^{(0)}$ or $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ be the contravariant stress tensor with components $\bar{\underset{\sim}{\sigma}}_{ij}^{(0)}$ or $\underset{\sim}{\sigma}_{ij}^{(0)}$ and dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$. Component $\bar{\underset{\sim}{\sigma}}_{11}^{(0)}$ or $\underset{\sim}{\sigma}_{11}^{(0)}$ is in the $\tilde{\boldsymbol{g}}^1$ direction on a face of the tetrahedron with unit exterior normal $\tilde{\boldsymbol{g}}^1$ i.e. on the $\tilde{\boldsymbol{g}}^1$ face. Likewise $\bar{\underset{\sim}{\sigma}}_{12}^{(0)}$ or $\underset{\sim}{\sigma}_{12}^{(0)}$ and $\bar{\underset{\sim}{\sigma}}_{31}^{(0)}$ or $\underset{\sim}{\sigma}_{31}^{(0)}$ act on the $\tilde{\boldsymbol{g}}^1$ and $\tilde{\boldsymbol{g}}^3$ faces in the $\tilde{\boldsymbol{g}}^2$ and $\tilde{\boldsymbol{g}}^1$ directions. Using the dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariance law of transformation we can write

$$\underset{\sim}{\boldsymbol{\sigma}}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{\sigma}_{ij}^{(0)} \tag{2.28}$$

using (2.22) we can write

$$\boldsymbol{\sigma}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \sigma_{ij}^{(0)} \quad ; \quad \sigma_{ij}^{(0)} = J_{ik} \underset{\sim}{\sigma}_{kl}^{(0)} J_{jl} \quad \text{or} \quad [\sigma^{(0)}]^T = [J]\,[\underset{\sim}{\sigma}^{(0)}]\,[J]^T \tag{2.29}$$

$\boldsymbol{\sigma}^{(0)}$ is the contravariant Cauchy stress tensor (Lagrangian) from which $\bar{\boldsymbol{\sigma}}^{(0)}$ can be easily obtained by replacing $[J]$ with $\left[\bar{J}\right]^{-1}$ and $\boldsymbol{\sigma}^{(0)}$ with $\bar{\boldsymbol{\sigma}}^{(0)}$ in (2.29). Since the dyads of $\boldsymbol{\sigma}^{(0)}$ or $\bar{\boldsymbol{\sigma}}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}^{(0)}$ i.e.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.30}$$

2.1.5.2 Covariant Cauchy stress tensor

Instead of using contravariant directions and stress components $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ and covariant basis $\tilde{\boldsymbol{g}}_i$ we could use covariant stress components $(\underset{\sim}{\sigma}_{(0)})_{ij}$ or $(\bar{\underset{\sim}{\sigma}}_{(0)})_{ij}$ and contravariant basis $\tilde{\boldsymbol{g}}^i$. Consideration of $(\underset{\sim}{\sigma}_{(0)})_{ij}$ of course will require a different deformed tetrahedron such that covariant vectors $\tilde{\boldsymbol{g}}_i$ are normal to its non-oblique faces. The adverse consequences of choosing this measure of stress for finite deformation are discussed in references [71, 75]. Here we proceed using this measure as an alternative to the contravariant stress measure. Using dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$ and components $(\underset{\sim}{\sigma}_{(0)})_{ij}$ we can write

$$\bar{\boldsymbol{\sigma}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j (\underset{\sim}{\sigma}_{(0)})_{ij} \tag{2.31}$$

using (2.24) in (2.31) we can write

$$\bar{\boldsymbol{\sigma}}_{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{\sigma}_{(0)}\right)_{ij} \quad ; \quad \left(\bar{\sigma}_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{\sigma}_{(0)}\right)_{kl} \bar{J}_{lj} \quad \text{or} \quad [\bar{\sigma}_{(0)}] = \left[\bar{J}\right]^T [\underset{\sim}{\sigma}_{(0)}][\bar{J}] \tag{2.32}$$

$\bar{\boldsymbol{\sigma}}_{(0)}$ is the covariant Cauchy stress tensor (Eulerian) from which $\boldsymbol{\sigma}_{(0)}$ can be obtained by replacing $[\bar{J}]$ with $[J]^{-1}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ with $\boldsymbol{\sigma}^{(0)}$ in (2.32). Since the dyads of $\bar{\boldsymbol{\sigma}}_{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ i.e.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.33}$$

Remark

The Cauchy stress tensors $\boldsymbol{\sigma}^{(0)}$ or $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\boldsymbol{\sigma}_{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ are nonsymmetric at this stage and so are stress tensors $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ and $\underset{\sim}{\boldsymbol{\sigma}}_{(0)}$. Following the details in reference [71] we can also define Jaumann stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$ using $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ stress measures.

2.1.6 Definitions of moment tensors

2.1.6.1 Contravariant Cauchy moment tensor

When the deformed tetrahedron with moment $\bar{\boldsymbol{M}}$ (per unit area) on its oblique face $\bar{A}\bar{B}\bar{C}$ is isolated from volume $\bar{V}$, its non-oblique face will have existence of moments (per unit area) on them. As in the case of stress, contravariant basis is the most natural way to define these. Let $\underset{\sim}{\boldsymbol{m}}^{(0)}$ or $\underset{\sim}{\bar{\boldsymbol{m}}}^{(0)}$ be the contravariant moment tensors with components $\underset{\sim}{m}_{ij}^{(0)}$ or $\underset{\sim}{\bar{m}}_{ij}^{(0)}$ and dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$. Component $\underset{\sim}{m}_{11}^{(0)}$ or $\underset{\sim}{\bar{m}}_{11}^{(0)}$ is along $\tilde{\boldsymbol{g}}^1$ direction on a face of the tetrahedron with unit exterior normal $\tilde{\boldsymbol{g}}^1$ i.e. on $\tilde{\boldsymbol{g}}^1$ face. Likewise $\underset{\sim}{m}_{12}^{(0)}$ or $\underset{\sim}{\bar{m}}_{12}^{(0)}$ and $\underset{\sim}{m}_{31}^{(0)}$ or $\underset{\sim}{\bar{m}}_{31}^{(0)}$ act on $\tilde{\boldsymbol{g}}^1$ and $\tilde{\boldsymbol{g}}^3$ faces in the $\tilde{\boldsymbol{g}}^2$ and $\tilde{\boldsymbol{g}}^1$ directions. Using the dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariance law of transformation we can write

$$\boldsymbol{m}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{m}_{ij}^{(0)} \tag{2.34}$$

Using (2.22) we can write

$$\boldsymbol{m}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j m_{ij}^{(0)} \quad ; \quad m_{ij}^{(0)} = J_{ik} \underset{\sim}{m}_{kl}^{(0)} J_{jl} \quad \text{or} \quad [m^{(0)}]^T = [J]\,[\underset{\sim}{m}^{(0)}]\,[J]^T \tag{2.35}$$

$\boldsymbol{m}^{(0)}$ is contravariant Cauchy moment tensor (Lagrangian) from which $\bar{\boldsymbol{m}}^{(0)}$ can be obtained by replacing $[J]$ with $[\bar{J}]^{-1}$ and $\boldsymbol{m}^{(0)}$ with $\bar{\boldsymbol{m}}^{(0)}$. Since the dyads of $\boldsymbol{m}^{(0)}$ or $\bar{\boldsymbol{m}}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, based on Koiter [10], the Cauchy principle is assumed to hold between $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{m}}^{(0)}$ i.e.

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.36}$$

We need to establish whether $\bar{\boldsymbol{m}}^{(0)}$ is symmetric or not, hence at this stage $\bar{\boldsymbol{m}}^{(0)}$ is not symmetric.

2.1.6.2 Covariant Cauchy moment tensor

Instead of using contravariant directions we could instead use covariant directions with moment tensor components $(\underset{\sim}{m}_{(0)})_{ij}$ and contravariant basis with dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$. Consideration of $(\underset{\sim}{m}_{(0)})_{ij}$

will of course require a different deformed tetrahedron such that covariant vectors $\tilde{\boldsymbol{g}}_i$ are normal to its non-oblique faces. The adverse consequences of choosing this measure are similar to those for the choice of $(\underset{\sim}{\sigma}_{(0)})_{ij}$ for the stress measure. Using the dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$ with components $(\underset{\sim}{m}_{(0)})_{ij}$ we can write

$$\bar{\boldsymbol{m}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j \left(\underset{\sim}{m}_{(0)}\right)_{ij} \tag{2.37}$$

Using (2.23) we can write

$$\bar{\boldsymbol{m}}_{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{m}_{(0)}\right)_{ij} \quad ; \quad \left(\bar{m}_{(0)}\right)_{ij} = \bar{J}_{ki} \left(\underset{\sim}{m}_{(0)}\right)_{kl} \bar{J}_{lj} \quad \text{or} \quad [\bar{m}_{(0)}] = \left[\bar{J}\right]^T \left[\underset{\sim}{m}_{(0)}\right] \left[\bar{J}\right] \tag{2.38}$$

$\bar{\boldsymbol{m}}_{(0)}$ is a covariant Cauchy moment tensor (Eulerian) from which $\boldsymbol{m}_{(0)}$ can be obtained by replacing $[\bar{J}]$ with $[J]^{-1}$ and $\bar{\boldsymbol{m}}_{(0)}$ with $\boldsymbol{m}_{(0)}$. Following Koiter and since the dyads of $\bar{\boldsymbol{m}}_{(0)}$ are $\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j$, the Cauchy principle holds between $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{m}}_{(0)}$ i.e.

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.39}$$

As in the case of the contravariant moment tensor, $\bar{\boldsymbol{m}}_{(0)}$ is also a non-symmetric Cauchy moment tensor in covariant basis unless established otherwise.

2.2 Conservation and balance laws

We remark that the polar continuum theory considered here incorporates new physics due to rates of rotations. This physics is absent in the currently used thermodynamic framework for isotropic, homogeneous fluent continua. This new physics due to rates of rotations may influence some or all conservation and balance laws. In order to determine the precise influence of the new physics (or lack of it) on the conservation and balance laws, we must initiate the derivations of the conservation and balance laws at a fundamental stage as we do for the non-polar case [71] so that the resulting equations can be compared with the non-polar case to determine how these laws are modified or influenced by the physics due to rates of rotations. We caution that after the derivation of conservation and balance laws we may find that some laws are not influenced by this new physics in which case the corresponding equations will obviously be the same as those for the non-polar case. Nonetheless the derivation of all conservation and balance laws must be presented in completeness otherwise we can not determine whether a particular law is influenced by this new physics when compared to the non-polar case.

In polar continuum theory we must consider the velocity gradient tensor and rate of rotation gradient tensor in the derivations of the following conservation and balance laws based on the assumption of thermodynamic equilibrium during evolution: (i) conservation of mass and conservation of inertia (ii) balance of linear momenta (iii) balance of angular momenta (iv) balance of moments of moments (v) first law of thermodynamics, conservation of energy (vi) second law of thermodynamics, entropy inequality. We present the derivations in the following.

2.2.1 Conservation of mass and inertia

The derivation of the continuity equation based on conservation of mass remains the same as for non-polar continuum, Following reference [71] we can derive the following continuity equation in Eulerian description.

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}} \cdot (\bar{\rho}\bar{\boldsymbol{v}}) = 0 \tag{2.40}$$

$$\text{or} \quad \frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\text{div}\,(\bar{\boldsymbol{v}}) = 0 \tag{2.41}$$

in which $\bar{\rho}(\bar{\boldsymbol{x}}, t)$ is the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration. Micro-polar continuum theories consider continua with micro-fibers. In a deforming volume of matter these micro-fibers (considered inextensible in micro-polar continuum theory) will have inertial effects due to rotation. Conservation of inertia refers to such inertial effects. In the polar continuum theory presented here this inertial effect is not present, due to the fact that we have not introduced a micro-continuum as part of the derivation of the conservation and balance laws. Thus, we assume that in the polar continuum theory considered here there is only one conservation law leading to the same continuity equation (2.40) or (2.41) as in the case of non-polar continuum theory.

2.2.2 Balance of linear momenta

For a deforming volume of matter, the rate change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. This derivation also is exactly the same as that for non-polar continuum theory. Following reference [71] we can write the following in Eulerian description (using contravariant Cauchy stress tensor).

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{2.42}$$

$$\text{or} \quad \bar{\rho}\frac{\partial \bar{v}_i}{\partial t} + \bar{\rho}\bar{v}_j\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial \bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} = 0 \tag{2.43}$$

in which $\bar{\boldsymbol{F}}^b$ are body forces per unit mass and $\bar{\boldsymbol{\sigma}}^{(0)}$ is the contravariant Cauchy stress tensor (See reference [71] for using covariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ and Jaumann stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$ in place of $\bar{\boldsymbol{\sigma}}^{(0)}$ and the consequences of doing so). Equations (2.42) or (2.43) are the momentum equations in x_1, x_2, and x_3 directions.

2.2.3 Balance of angular momenta

The principle of balance of angular momenta for a polar continuum can be stated as follows. The time rate of change of total moment of momentum for a polar continuum is equal to the vector sum of the moments of external forces and the moments. Thus, due to the surface stress $\bar{\boldsymbol{P}}$, surface moment $\bar{\boldsymbol{M}}$ (per unit area), body force $\bar{\boldsymbol{F}}^b$ (per unit mass), and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using Eulerian description) we can write the following

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} \tag{2.44}$$

In the following derivation we consider contravariant basis. We use Cauchy principle $\bar{\boldsymbol{P}} = (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\boldsymbol{n}}$ or $\bar{P}_j = \bar{\sigma}^{(0)}_{mj}\bar{n}_m$ and express cross products using permutation symbol $\boldsymbol{\epsilon}$. We also use Cauchy principle $\bar{\boldsymbol{M}} = (\bar{\boldsymbol{m}}^{(0)})^T \cdot \bar{\boldsymbol{n}}$ or $\bar{M}_k = \bar{m}^{(0)}_{mk}\bar{n}_m$. Substituting into (2.44).

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{v}_j d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\epsilon_{ijk}\bar{x}_i\bar{\sigma}^{(0)}_{mj}\bar{n}_m - \bar{m}^{(0)}_{mk}\bar{n}_m\right) d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{F}^b_j d\bar{V} \tag{2.45}$$

Using transport theorem for the left side of (2.45), Gauss's divergence theorem for the first term on the right side of (2.45) and using $\frac{D\bar{x}_i}{Dt} = \bar{v}_i$

$$\int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\left(\bar{v}_i\bar{v}_j + \bar{x}_i\frac{D\bar{v}_j}{Dt}\right) d\bar{V} = \int\limits_{\bar{V}(t)} \left(\epsilon_{ijk}\left(\bar{x}_i\bar{\sigma}^{(0)}_{mj}\right)_{,m} - \left(\bar{m}^{(0)}_{mk}\right)_{,m}\right) d\bar{V} + \int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{F}^b_j d\bar{V} \tag{2.46}$$

We note that

$$\epsilon_{ijk}\bar{v}_i\bar{v}_j = 0 \tag{2.47}$$

and

$$\begin{aligned}\left(\bar{x}_i\bar{\sigma}^{(0)}_{mj}\right)_{,m} &= \bar{x}_{i,m}\bar{\sigma}^{(0)}_{mj} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m} \\ &= \delta_{im}\bar{\sigma}^{(0)}_{mj} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m} \\ &= \bar{\sigma}^{(0)}_{ij} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m}\end{aligned} \tag{2.48}$$

Using (2.47) and (2.48) in (2.46) and regrouping

$$\int\limits_{\bar{V}(t)} \epsilon_{ijk}\left(\bar{x}_i\left(\bar{\rho}\frac{D\bar{v}_j}{Dt} - \bar{\rho}\bar{F}^b_j - \bar{\sigma}^{(0)}_{mj,m}\right)\right) d\bar{V} = \int\limits_{\bar{V}(t)} \left(-\bar{m}^{(0)}_{mk,m} + \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij}\right) d\bar{V} \tag{2.49}$$

Using momentum equations (2.43) in (2.49), we obtain

$$\int\limits_{\bar{V}(t)} \left(-\bar{m}^{(0)}_{mk,m} + \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij}\right) d\bar{V} = 0 \tag{2.50}$$

Since $\bar{V}(t)$ is arbitrary, (2.49) implies

$$\bar{m}^{(0)}_{mk,m} - \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij} = 0 \tag{2.51}$$

Equations (2.51) represents balance of angular momenta. We note that $\bar{\boldsymbol{\sigma}}^{(0)}$ is a nonsymmetric

Cauchy stress tensor. It is instructive to expand (2.51) into three equations

$$\begin{aligned}
\frac{\partial \bar{m}^{(0)}_{i1}}{\partial \bar{x}_i} - \left(\bar{\sigma}^{(0)}_{23} - \bar{\sigma}^{(0)}_{32}\right) = 0 \\
\frac{\partial \bar{m}^{(0)}_{i2}}{\partial \bar{x}_i} - \left(\bar{\sigma}^{(0)}_{31} - \bar{\sigma}^{(0)}_{13}\right) = 0 \\
\frac{\partial \bar{m}^{(0)}_{i3}}{\partial \bar{x}_i} - \left(\bar{\sigma}^{(0)}_{12} - \bar{\sigma}^{(0)}_{21}\right) = 0
\end{aligned} \tag{2.52}$$

From (2.52), we note that the off diagonal elements of stress tensor $\bar{\boldsymbol{\sigma}}^0$ are balanced by the gradients of the Cauchy moment tensor. Equations (2.52) can also be obtained in covariant basis and Jaumann rates by replacing $\bar{\boldsymbol{m}}^{(0)}, \bar{\boldsymbol{\sigma}}^{(0)}$ with $\bar{\boldsymbol{m}}_{(0)}, \bar{\boldsymbol{\sigma}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{m}}^J, {}^{(0)}\bar{\boldsymbol{\sigma}}^J$.

Remarks

1. In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components obviously do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by shear stresses.

2. In the case of non-polar fluent continua, the balance of angular momenta is a statement of self equilibrating moments due to shear stresses that yields

$$\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{2.53}$$

   which implies that $\bar{\boldsymbol{\sigma}}^{(0)}$ is symmetric. An important point to note is that (2.53) is a result of stress couples due to shear stresses.

3. In the case of polar continua, the existence of moments $[\bar{m}^{(0)}]$ due to the material constitution resisting the rotations results in the shear stress couples being balanced by the internal moments. Thus, for polar continua, the balance of angular momenta yields (2.52) instead of (2.53), i.e.

$$[\bar{m}^{(0)}]^T \left\{\bar{\nabla}\right\} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{2.54}$$

   We note that (2.54) is also a result of stress couples caused by shear stresses.

4. Thus, both non-polar and polar continuum theories use stress couples in the angular momenta balance law. *Referring to the polar continuum theory presented here as stress couple theory is inappropriate as the non-polar theory also make use of stress couples.*

5. From (2.51) or (2.52) we note that gradients of $[\bar{m}^{(0)}]$ equilibrate with the antisymmetric components of the stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ as the symmetric components cancel each other in each of the three equations in (2.52).

6. The derivation of the balance of angular momenta presented here does not include any externally applied body couples. While the inclusion of applied body couples in this derivation would be a simple matter of including an additional term similar to $\boldsymbol{F}^b$ in the balance of linear momenta, the result of having a non-symmetric stress tensor remains the same. That is, *it is not sufficient to say that the symmetry of the stress tensor is due to the absence of body couples*, but additionally it requires that the continua provides no resistance to varying rates of rotations. If externally applied body couples are included, then the resulting theory should not be called an "internal polar theory".

7. Lastly, we emphasize that *appearance of equation (2.51) in other theories published in the literature does not necessarily make the polar continuum theory presented here the same as those in the literature.* In this work, we begin by demonstrating that the varying rotation rates at neighboring locations, when resisted by the deforming fluent continua, require existence of internal moment tensor $[\bar{m}^{(0)}]$. The balance of angular momenta establishes a relationship between $[\bar{m}^{(0)}]$ and $[\bar{\sigma}^{(0)}]$ (equations (2.51) or 2.52).

2.2.4 Balance of moments of the moments (or couples)

Forces, moments, moments of moments ... are progressively higher order effects or terms, hence must satisfy appropriate balance laws to ensure absence of rigid rotation or rigid translation of the deforming volume of continua. Balance of angular momenta (moments of forces) must be considered for couples created by forces and the moments. Likewise, since moment is similar to force, but is a higher order effect or term than force, a balance law similar to balance of angular momentum i.e. balance of moment of couples or moments must be considered to ensure lack of rigid motion of the deforming continua. Just like in the case of non-polar, isotropic, homogeneous fluent continua balance of angular momenta (moments of forces) restricts the Cauchy stress tensor to be symmetric, we can expect this balance law to impose some restrictions on the Cauchy moment tensor. Yang et al. [69] presents a similar argument in what is called "modified couple stress theory", based on the observation that in a polar continuum moments are not free vectors and therefore create their own higher order couples which much be balanced by a free vector (moment of couples).

For the deformed tetrahedron to be in equilibrium the moments of the moments (or couples) must vanish. In the moments of the moments we must consider $\bar{\boldsymbol{M}}$ and also shear components of $\bar{\boldsymbol{\sigma}}^{(0)}$ i.e. $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}$ (in contravariant basis). Thus, we can write

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} = 0 \tag{2.55}$$

We expand the second term in (2.55) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using divergence theorem.

$$
\begin{aligned}
\int\limits_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} &= \int\limits_{\partial\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{M}_j = \int\limits_{\partial\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{m}_{mj}^{(0)}\bar{n}_m d\bar{A} \\
&= \int\limits_{\bar{V}} \left(\epsilon_{ijk}\bar{x}_i\bar{m}_{mj}^{(0)}\right)_{,m} d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk}\left(\bar{x}_{i,m}\bar{m}_{mj}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk}\left(\delta_{im}\bar{m}_{mj}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk}\left(\bar{m}_{ij}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} + \int\limits_{\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{m}_{mj,m}^{(0)} d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} + \int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{m}}^{(0)} \cdot \bar{\boldsymbol{\nabla}}\right) d\bar{V}
\end{aligned}
\tag{2.56}
$$

Using (2.56) in (2.55) and collecting terms

$$
\int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{m}}^{(0)} \cdot \bar{\boldsymbol{\nabla}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) d\bar{V} - \int\limits_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{2.57}
$$

The first term in (2.57) vanishes due to (2.51) (balance of angular momenta) and we obtain

$$
\int\limits_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{2.58}
$$

Since $\bar{V}$ is arbitrary, (2.58) implies

$$
\epsilon_{ijk}\bar{m}_{ij}^{(0)} = 0 \tag{2.59}
$$

That is $\bar{m}_{ij}^{(0)}$, the Cauchy moment tensor, is symmetric. Relation (2.59) also holds in covariant basis and Jaumann rates by replacing $\bar{\boldsymbol{m}}^{(0)}$ with $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{m}}^{J}$. Thus, we can see that the consequence of this balance law is to impose the restriction of symmetry on the Cauchy moment tensor.

We note that in the polar theory presented here, the Cauchy moment tensor is symmetric, but the Cauchy stress tensor is nonsymmetric, whereas in the corresponding non-polar theory, Cauchy stress tensor is symmetric and Cauchy moment tensor is null as rates of rotations are ignored in the theory. Symmetry of the Cauchy moment tensor is a restriction placed on the Cauchy moment tensor due to this balance law.

2.2.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in the increase in energy of the system. Expressing this as a rate statement we can write [71]

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{2.60}$$

$\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \tag{2.61}$$

$$\frac{D\bar{Q}}{Dt} = -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}d\bar{A} \tag{2.62}$$

$$\frac{D\bar{W}}{Dt} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}\right)d\bar{A} \tag{2.63}$$

Where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force per unit mass, $\bar{\boldsymbol{u}}$ are displacement, and $\bar{\boldsymbol{q}}$ is rate of heat. Note the additional term $\bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of rotation. In (2.61), we have not included energy due to rotary inertia. This is consistent with the assumption used in the conservation law in section 2.2.1. We expand each of the integrals in (2.61)–(2.63). Following reference [71], it is straight forward to show that:

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} = \int\limits_{\bar{V}(t)} \bar{\rho}\left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}}\cdot\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{v}}\right)d\bar{V} \tag{2.64}$$

$$-\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}d\bar{A} = -\int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}d\bar{V}; \quad \text{divergence theorem} \tag{2.65}$$

$$\frac{D\bar{W}}{Dt} = \int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}}d\bar{A} + \int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}d\bar{A} \tag{2.66}$$

Using contravariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$, Cauchy principle, and following the details in reference [71] we can write

$$\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}}d\bar{A} = \int\limits_{\bar{V}(t)} \left(\bar{\boldsymbol{v}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\sigma}}^{(0)}\right) + \bar{\sigma}_{ji}^{(0)}\frac{\partial\bar{v}_i}{\partial\bar{x}_j}\right)d\bar{V} \tag{2.67}$$

Likewise using contravariant moment tensor (per unit area) $\bar{\boldsymbol{m}}^{(0)}$, Cauchy principle, and following the details similar to these used in deriving (2.67), we can write

$$\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{M}} \cdot {}^t\bar{\boldsymbol{\Theta}} d\bar{A} = \int\limits_{\bar{V}(t)} \left({}^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}^{(0)} \right) + \bar{m}_{ji}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} \right) d\bar{V} \tag{2.68}$$

Using (2.64)–(2.68) in (2.60)

$$\begin{aligned} \int\limits_{\bar{V}(t)} \bar{\rho} \left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}} \cdot \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{v}} \right) d\bar{V} = & - \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} d\bar{V} + \int\limits_{\bar{V}(t)} \left(\bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}}^{(0)} \right) + \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) d\bar{V} \\ & + \int\limits_{\bar{V}(t)} \left({}^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}^{(0)} \right) + \bar{m}_{ji}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} \right) d\bar{V} \end{aligned} \tag{2.69}$$

Transferring all terms to the left of the equality and regrouping

$$\begin{aligned} & \int\limits_{\bar{V}(t)} \bar{\rho} \left(\bar{\boldsymbol{v}} \cdot \left(\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}}^{(0)} \right) \right) d\bar{V} \\ & \quad + \int\limits_{\bar{V}(t)} \left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}^{(0)} \right) \right) d\bar{V} = 0 \end{aligned} \tag{2.70}$$

Using (2.42) (balance of linear momenta) and (2.51) balance of angular momenta, (2.69) reduces to

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} \right) \right) d\bar{V} = 0 \tag{2.71}$$

Since $\bar{V}(t)$ is arbitrary, (2.71) implies that

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} \right) = 0 \tag{2.72}$$

Equation (2.72) is the final form of the energy equation in which $\bar{\boldsymbol{\sigma}}^{(0)}$ is a nonsymmetric Cauchy stress tensor and $\bar{\boldsymbol{m}}^{(0)}$ is a symmetric Cauchy moment tensor. Thus in (2.72) we can use

$$\bar{m}_{mj}^{(0)} \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} = \bar{m}_{mj}^{(0)} \left({}^{\bar{\Theta}}\bar{D}_{ij} + {}^{\bar{\Theta}}\bar{W}_{ij} \right) = \bar{m}_{mj}^{(0)} \left({}^{\bar{\Theta}}\bar{D}_{ij} \right); \quad \text{as} \quad \bar{m}_{mj}^{(0)} \left({}^{\bar{\Theta}}\bar{W}_{ij} \right) = 0 \tag{2.73}$$

Equation (2.72) representing balance of energy can also be derived in covariant basis or in Jaumann rates. In (2.72) we replace $\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$ by $\bar{\boldsymbol{\sigma}}_{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$, ${}^{(0)}\bar{\boldsymbol{m}}^J$ to obtain its corresponding form in covariant basis and in Jaumann rates.

2.2.6 Second law of thermodynamics

If $\bar{\eta}$ is the entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non-contacting bodies, then the rate of increase in entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [71, 73, 74]. Thus

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}d\bar{V} \geq \int\limits_{\partial\bar{V}(t)} \bar{h}d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}d\bar{V} \tag{2.74}$$

Using Cauchy's postulate for $\bar{h}$ i.e.

$$\bar{h} = -\bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}} \tag{2.75}$$

Using (2.75) in (2.74)

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}d\bar{V} \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}}d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}d\bar{V} \tag{2.76}$$

We recall that [71]

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}d\bar{V} = \int\limits_{\bar{V}(t)} \bar{\rho}\frac{D\bar{\eta}}{Dt}d\bar{V} \tag{2.77}$$

and

$$-\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}}d\bar{A} = -\int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\Psi}}d\bar{V} = -\int\limits_{\bar{V}(t)} \bar{\Psi}_{i,i}d\bar{V}; \quad \text{divergence theorem} \tag{2.78}$$

Substituting from (2.77) and (2.78) in (2.76) and transferring all terms to the left of inequality

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\Psi}_{i,i} - \bar{s}\bar{\rho}\right) d\bar{V} \geq 0 \tag{2.79}$$

Since volume $\bar{V}(t)$ is arbitrary, (2.79) implies

$$\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\Psi}_{i,i} - \bar{s}\bar{\rho} \geq 0 \tag{2.80}$$

Equation (2.80) is the entropy inequality and is the most fundamental form resulting from the second law of thermodynamics. A more useful form of (2.80) can be derived if we assume

$$\bar{\boldsymbol{\Psi}} = \frac{\bar{\boldsymbol{q}}}{\bar{\theta}} \quad ; \quad \bar{s} = \frac{\bar{r}}{\bar{\theta}} \tag{2.81}$$

Where $\bar{\theta}$ is absolute temperature, $\bar{\boldsymbol{q}}$ is heat vector, and $\bar{r}$ is a suitable potential. Using (2.81)

$$\bar{\Psi}_{i,i} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i}{(\bar{\theta})^2}\bar{\theta}_{,i} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i}{(\bar{\theta})^2}\bar{g}_i \quad ; \quad \bar{g}_i = \bar{\theta}_{,i} \tag{2.82}$$

Substituting for $\bar{s}$ from (2.81) and for $\bar{\Psi}_{i,i}$ from (2.82) into (2.80) and multiplying by $\bar{\theta}$.

$$\bar{\rho}\theta\frac{D\bar{\eta}}{Dt} + (\bar{q}_{i,i} - \bar{\rho}\bar{r}) - \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} \geq 0 \tag{2.83}$$

From energy equation (2.72) (after inserting $\bar{\rho}\bar{r}$ term) in contravariant basis

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \bar{\rho}\bar{r} = \bar{q}_{i,i} - \bar{\rho}\bar{r} = -\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\sigma}_{ji}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + \bar{m}_{ji}^{(0)}\frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} + {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) \tag{2.84}$$

Substituting from (2.84) into (2.83)

$$\bar{\rho}\theta\frac{D\bar{\eta}}{Dt} - \bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\sigma}_{ji}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + \bar{m}_{ji}^{(0)}\frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} + {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) - \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} \geq 0 \tag{2.85}$$

or

$$\bar{\rho}\left(\frac{D\bar{e}}{Dt} - \theta\frac{D\bar{\eta}}{Dt}\right) - \bar{\sigma}_{ji}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)}\frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) + \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} \leq 0 \tag{2.86}$$

Let $\bar{\Phi}$ be Helmholtz free energy density (specific Helmholtz free energy) defined by

$$\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta} \tag{2.87}$$

Hence

$$\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt} = \left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) \tag{2.88}$$

Substituting from (2.88) into (2.86)

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - \bar{\sigma}_{ji}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)}\frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) \leq 0 \tag{2.89}$$

or

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - \operatorname{tr}\left([\bar{\sigma}^{(0)}]^T[\bar{L}]^T\right) - \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{L}]\right) - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) \leq 0 \tag{2.90}$$

$\bar{\boldsymbol{m}}^{(0)}$ is symmetric but $\bar{\boldsymbol{\sigma}}^{(0)}$ is not symmetric. Since $\bar{\boldsymbol{m}}^{(0)}$ is symmetric, we can use the following in (2.90).

$$\operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{L}]\right) = \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{D}]\right) \tag{2.91}$$

The entropy inequality (2.90) in covariant basis and in Jaumann rates can be obtained by replacing $\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$ with $\bar{\boldsymbol{\sigma}}_{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$, ${}^{(0)}\bar{\boldsymbol{m}}^J$.

2.2.7 Stress decomposition and balance laws

It is instructive to decompose stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ (considering contravariant basis) into symmetric ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ and antisymmetric ${}_a\bar{\boldsymbol{\sigma}}^{(0)}$ tensors

$$\bar{\boldsymbol{\sigma}}^{(0)} =_s \bar{\boldsymbol{\sigma}}^{(0)} +_a \bar{\boldsymbol{\sigma}}^{(0)} \tag{2.92}$$

where

$$\begin{aligned} {}_s\bar{\boldsymbol{\sigma}}^{(0)} &= \frac{1}{2}\left(\bar{\boldsymbol{\sigma}}^{(0)} + \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T\right) \\ {}_a\bar{\boldsymbol{\sigma}}^{(0)} &= \frac{1}{2}\left(\bar{\boldsymbol{\sigma}}^{(0)} - \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T\right) \end{aligned} \tag{2.93}$$

We substitute these in the balance of linear momenta (2.43), balance of angular momenta (2.51), energy equation (2.72), and entropy inequality (2.90). First we note that

$$\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = \boldsymbol{\epsilon} : \left({}_s\bar{\boldsymbol{\sigma}}^{(0)} +_a \bar{\boldsymbol{\sigma}}^{(0)}\right) = \boldsymbol{\epsilon} : \left({}_a\bar{\boldsymbol{\sigma}}^{(0)}\right) \tag{2.94}$$

as

$$\boldsymbol{\epsilon} : \left({}_s\bar{\boldsymbol{\sigma}}^{(0)}\right) = 0 \tag{2.95}$$

$$\bar{\sigma}_{ji}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} = \left({}_s\bar{\sigma}_{ji}^{(0)} +_a \bar{\sigma}_{ji}^{(0)}\right)\left(\bar{D}_{ij} + \bar{W}_{ij}\right) = \left({}_s\bar{\sigma}_{ji}^{(0)}\right)\bar{D}_{ij} + \left({}_a\bar{\sigma}_{ji}^{(0)}\right)\bar{W}_{ij} \tag{2.96}$$

as

$$\left({}_s\bar{\sigma}_{ji}^{(0)}\right)\bar{W}_{ij} = \left({}_a\bar{\sigma}_{ji}^{(0)}\right)\bar{D}_{ij} = 0 \tag{2.97}$$

we can write (2.96) as

$$\mathrm{tr}\left([\bar{\sigma}^{(0)}][\bar{L}]\right) = \mathrm{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}]\right) + \mathrm{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}]\right) \tag{2.98}$$

Using (2.94)–(2.98) in (2.43), (2.51), (2.72), and (2.90) we can obtain

$$\bar{\rho}\frac{\partial \bar{v}_i}{Dt} + \bar{\rho}\bar{v}_j\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial_s\bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} - \frac{\partial_a\bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} = 0 \tag{2.99}$$

$$\bar{m}_{mk,m}^{(0)} + \epsilon_{ijk}\left({}_a\bar{\sigma}_{ij}^{(0)}\right) = 0 \tag{2.100}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \mathrm{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}]\right) - \mathrm{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}]\right) - \mathrm{tr}\left([\bar{m}^{(0)}][{}^{\Theta}\bar{D}]\right) - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}_a\bar{\boldsymbol{\sigma}}^{(0)}\right) = 0 \tag{2.101}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{q}_i\bar{g}_i}{\bar{\theta}}-\operatorname{tr}\left([_s\bar{\sigma}^{(0)}][\bar{D}]\right)-\operatorname{tr}\left([_a\bar{\sigma}^{(0)}][\bar{W}]\right)-\operatorname{tr}\left([\bar{m}^{(0)}][^{\bar{\Theta}}\bar{D}]\right)-{}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}_a\bar{\boldsymbol{\sigma}}^{(0)}\right)\leq 0 \tag{2.102}$$

A simple calculation by expanding the terms shows that

$$\operatorname{tr}\left([_a\bar{\sigma}^{(0)}][\bar{W}]\right)=-{}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}_a\bar{\boldsymbol{\sigma}}^{(0)}\right) \tag{2.103}$$

If we substitute (2.103) in (2.101) and (2.102) then the energy equation and entropy inequality simplify.

$$\bar{\rho}\frac{D\bar{e}}{Dt}+\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}-\operatorname{tr}\left([_s\bar{\sigma}^{(0)}][\bar{D}]\right)-\operatorname{tr}\left([\bar{m}^{(0)}][^{\bar{\Theta}}\bar{D}]\right)=0 \tag{2.104}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{q}_i\bar{g}_i}{\bar{\theta}}-\operatorname{tr}\left([_s\bar{\sigma}^{(0)}][\bar{D}]\right)-\operatorname{tr}\left([\bar{m}^{(0)}][^{\bar{\Theta}}\bar{D}]\right)\leq 0 \tag{2.105}$$

Remarks

(1) Equations (2.99), (2.100), (2.104), and (2.105) can also be expressed in covariant basis and using Jaumann rates.

(2) Equations (2.40), (2.99), (2.100), (2.104), and (2.105) constitute a complete mathematical model for fluent media in Eulerian description.

(3) From (2.104) and (2.105) we can conclude that ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{m}}^{(0)}$, ${}^{\bar{\Theta}}\bar{\boldsymbol{D}}$ are conjugate pairs, hence are responsible for conversion of mechanical energy into heat or entropy. The conjugate pairs are instrumental in deciding the dependent variables in the constitutive theories and some of their argument tensors. These conjugate pairs suggest that ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ can be expressed as a function of $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{m}}^{(0)}$ as a function of ${}^{\bar{\Theta}}\bar{\boldsymbol{D}}$. We note that $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ are also conjugate, thus $\bar{\boldsymbol{q}}$ can be expressed as a function of $\bar{\boldsymbol{g}}$. These details will be considered in the following sections.

2.3 Dependent variables in the constitutive theories and their argument tensors

In section 2.2, conservation and balance laws are derived for internal polar fluent media. These derivations were presented using contravariant and covariant measures of stress tensor and moment tensor as well as using Jaumann rates. Measures of stress, moment, and strain tensors and their convected time derivatives can be considered in contravariant basis, covariant basis, or Jaumann rates. Following reference [71] for example $(\bar{\boldsymbol{\sigma}}^{(0)},\bar{\boldsymbol{\sigma}}_{(0)},{}^{(0)}\bar{\boldsymbol{\sigma}}^J)$, $(\bar{\boldsymbol{m}}^{(0)},\bar{\boldsymbol{m}}_{(0)},{}^{(0)}\bar{\boldsymbol{m}}^J)$ can be considered as measures of Cauchy stress and moment tensors in contravariant and covariant basis and corresponding to Jaumann rates. Likewise we can let $[\gamma^{(k)}],[\gamma_{(k)}],[{}^{(k)}\gamma^J]$; $k=0,1,2,\dots n$ be the convected time derivatives of the Almansi, Green's strain tensor and Jaumann rate. Where, $[\gamma^{(0)}]=[\gamma^{(1)}]=[\gamma_{(0)}]=[\gamma_{(1)}]=[{}^{(0)}\gamma^J]=[{}^{(1)}\gamma^J]=[\bar{D}]$, symmetric part of the velocity gradient tensor. Let ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $[{}^{(k)}\gamma]$; $k=0,1,\dots,n$ define Cauchy stress tensor, Cauchy moment tensor and convected time derivatives of conjugate strain tensor in a chosen basis. We present derivations of the constitutive theories using this notation so that the resulting derivations are basis independent. By replacing $({}^{(0)}\bar{\boldsymbol{\sigma}},\ {}^{(0)}\bar{\boldsymbol{m}},\ [{}^{(k)}\gamma];\ k=0,1,\dots,n)$ with $(\bar{\boldsymbol{\sigma}}^{(0)},\ \bar{\boldsymbol{m}}^{(0)},\ [\gamma_{(k)}];\ k=0,1,\dots,n)$, $(\bar{\boldsymbol{\sigma}}_{(0)},\ \bar{\boldsymbol{m}}_{(0)},\ [\gamma^{(k)}];\ k=0,1,\dots,n)$, and $({}^{(0)}\bar{\boldsymbol{\sigma}}^J,\ {}^{(0)}\bar{\boldsymbol{m}}^J,\ [{}^{(k)}\gamma^J];\ k=0,1,\dots,n)$, the constitutive theories in contravariant basis, covariant basis, and in Jaumann rates can be obtained. In addition to the convected time derivatives of the strain tensor, we must also consider the rate of rotation gradient tensor in the derivation of the constitutive theory for internally polar thermofluids. In section 2.2 we show that the Cauchy moment tensor and symmetric part of the rate of rotation gradient tensors are conjugate. Thus, dependence of the Cauchy moment tensor on the symmetric part of the gradient of rate of rotation tensor must be considered in addition to its other argument tensors. Whether the resulting constitutive theories for the Cauchy moment tensor are basis dependent or not finally depends on the basis dependency (or lack of it) of its argument tensors. In section 2.1.6 it is shown that the Cauchy moment tensor is certainly basis dependent thus at the onset of the derivation of the constitutive theory we consider ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$, the Cauchy stress and moment tensors and heat vector in a chosen basis i.e. the convected time derivatives of order zero of the corresponding stress, moment, and heat tensors in a chosen basis as the dependent variables in the constitutive theories.

Using the basis independent notations for Cauchy stress and moment tensor and heat vector and the convected time derivatives of the conjugate strain tensor we can write the following for the conservation and balance laws derived and presented in section 2.2, in the absence of conservation of inertia, sources and sinks, and entropy due to non-contacting sources in the energy equation. The conservation and balance laws for internal polar thermofluids, after many simplifications shown is section 2.2, now yield (2.40), (2.99), (2.51), (2.59), (2.104), (2.105) in which ${}^s\psi_d$ and ${}^m\psi_d$, the terms in the energy equation that are responsible for dissipation of mechanical work into heat, are defined by (2.112). We rewrite these in the following convenient form.

$$\frac{\partial\bar{\rho}}{\partial t}+\bar{\boldsymbol{\nabla}}\cdot(\bar{\rho}\bar{\boldsymbol{v}})=0 \tag{2.106}$$

$$\bar{\rho}\frac{\partial\bar{v}_i}{\partial t}+\bar{\rho}\bar{v}_j\frac{\partial\bar{v}_i}{\partial\bar{x}_j}-\frac{\partial({}_s^{(0)}\bar{\sigma}_{ji})}{\partial\bar{x}_j}-\frac{\partial({}_a^{(0)}\bar{\sigma}_{ji})}{\partial\bar{x}_j}=0 \tag{2.107}$$

$${}^{(0)}\bar{m}_{pk,p}-\epsilon_{ijk}({}_a^{(0)}\bar{\sigma}_{ij})=0 \tag{2.108}$$

$$\epsilon_{ijk}{}^{(0)}\bar{m}_{ij}=0 \tag{2.109}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt}-\bar{\boldsymbol{\nabla}}\cdot{}^{(0)}\bar{\boldsymbol{q}}-{}^s\psi_d-{}^m\psi_d=0 \tag{2.110}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{e}}{Dt}\right)+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}^s\psi_d-{}^m\psi_d\le 0 \tag{2.111}$$

$$
{}^{s}\psi_d = \mathrm{tr}\left([{}_{s}^{(0)}\bar{\sigma}][{}^{(1)}\gamma] \right), \quad {}^{m}\psi_d = \mathrm{tr}\left([{}^{(0)}\bar{m}][{}^{\Theta}D] \right) \tag{2.112}
$$

The choice of dependent variables in the constitutive theories must be consistent with the axiom of casualty [71, 76]. The self observable quantities and those that can be derived from them by simple differentiation and/or integration can not be considered as dependent variables in the constitutive theories. Thus velocities, temperatures, temperature gradients, etc. are ruled out as choices of dependent variables in the constitutive theories. From the entropy inequality we note that ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{m}}, \bar{\Phi}, \bar{\eta}, {}^{(0)}\bar{\boldsymbol{q}}$ are possible choices of dependent variables in the constitutive theories. The choice of ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ as dependent variables in the constitutive theories is also supported by balance of linear momenta, balance of angular momenta, and the energy equation. ${}_{a}^{(0)}\bar{\boldsymbol{\sigma}}$ can not be chosen as dependent variables in the constitutive theories as these are deterministic from the balance of angular momenta. Choice of $\bar{e}, \bar{\eta}$ or $\bar{\Phi}, \bar{\eta}$ is a matter of preference as these are related through $\bar{\Phi}$. In the present work we choose $\bar{\Phi}, \bar{\eta}$, hence $\bar{e}$ need not be considered as a dependent variable in the constitutive theories. Thus, ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{m}}, \bar{\Phi}, \bar{\eta}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ are the possible dependent variables in the constitutive theories. At a later stage of the derivation, some of these may be ruled out as dependent variables in the constitutive theories if so warranted by some other considerations.

Next we consider possible choices of argument tensors of dependent variables, keeping in mind the principle of equipresence [71, 76], i.e. at the onset all dependent variables in the constitutive theories possibly must contain the same argument tensors. For compressible fluent media, density $\bar{\rho}$ is certainly an argument tensor. $\bar{\theta}$ is a natural choice as an argument tensor. The choice of $\bar{\boldsymbol{g}}$ as an argument tensor is necessitated due to the dependent variable ${}^{(0)}\bar{\boldsymbol{q}}$ in the constitutive theory and the physics of heat conduction. The choice of $[{}^{(1)}\gamma]$ and $[{}^{\Theta}D]$ as argument tensors is also clear as these are conjugate to ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$. From conservation of mass in Lagrangian description we know that $\rho_0 = |J|\rho$ i.e. compressibility is due to $|J| = \rho_0/\rho$, hence it is fitting to consider $1/\bar{\rho}$ as an argument tensor in Eulerian description as opposed to $\bar{\rho}$ for the dependent variables in the constitutive theories. At a later stage dependence on $1/\bar{\rho}$ can be replaced by dependence on $\bar{\rho}$ by using calculus. Thus, based on the principle of equipresence [71, 76] we have

$$
\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, [{}^{(1)}\gamma], [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, [{}^{(1)}\gamma], [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}_{s}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, [{}^{(1)}\gamma], [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, [{}^{(1)}\gamma], [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, [{}^{(1)}\gamma], [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{2.113}
$$

We note $[{}^{(1)}\gamma]$ is the first convected time derivative of the strain tensor (Almansi tensor or Green's tensor or Jaumann rates) and is a fundamental kinematic tensor. In addition to $[{}^{(1)}\gamma]$, we also have $[{}^{(k)}\gamma]$; $k = 2, \ldots, n$, as the fundamental kinematic tensors up to order n that are convected time derivatives of orders $2, 3, \dots, n$ of strain tensor in a chosen basis. With the choice of $[{}^{(1)}\gamma]$, the first convected time derivative of the strain tensor only in (2.114) the resulting constitutive theories would be rate constitutive theories of order one. We replace $[{}^{(1)}\gamma]$ with $[{}^{(k)}\gamma]$; $k = 1, 2, \dots, n$ as these all are fundamental kinematic tensors to generalize the derivation for the rate constitutive theories to up to order n.

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}_{s}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{2.114}$$

From the entropy inequality we note that $([{}_{s}^{(0)}\bar{\sigma}], [{}^{(1)}\gamma])$ and $([{}^{(0)}\bar{m}], [{}^{\Theta}D])$ are conjugate pairs i.e. $[{}_{s}^{(0)}\bar{\sigma}]$ has no dependence on $[{}^{\Theta}D]$ and likewise $[{}^{(0)}\bar{m}]$ has no dependence on $[{}^{(1)}\gamma]$. Thus we can modify the argument tensors of ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ in (2.114).

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}_{s}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, [{}^{(k)}\gamma];\ k = 1, 2, \dots, n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, [{}^{\Theta}D], \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{2.115}$$

2.4 Entropy inequality

Consider the entropy inequality (2.111) with the arguments of $\bar{\Phi}$ defined in (2.115). We now can obtain $\frac{D\Phi}{Dt} = \dot{\bar{\Phi}}$ needed in the entropy inequality

$$\dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial(\frac{1}{\bar{\rho}})}\left(-\frac{1}{\bar{\rho}^2}\right)\dot{\bar{\rho}} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial({}^{(j)}\gamma_{ik})}\left({}^{(j)}\dot{\gamma}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial({}^{\Theta}\bar{D}_{ik})}\left({}^{\Theta}\dot{\bar{D}}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} \tag{2.116}$$

From the continuity equation (2.106) (its alternate from in $\frac{D\bar{\rho}}{Dt} = \dot{\bar{\rho}}$)

$$\dot{\bar{\rho}} = -\bar{\rho}\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}^{(1)}\gamma_{kk} = -\bar{\rho}^{(1)}\gamma_{ik}\delta_{ik} \tag{2.117}$$

Using (2.117) in (2.116)

$$\dot{\bar{\Phi}} = -\frac{1}{\bar{\rho}} \frac{\partial \bar{\Phi}}{\partial(\frac{1}{\bar{\rho}})} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial({}^{(j)}\gamma_{ik})} \left({}^{(j)}\dot{\gamma}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial({}^{\bar{\Theta}}\bar{D}_{ik})} \left({}^{\bar{\Theta}}\dot{\bar{D}}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} \tag{2.118}$$

We note that

$$-\frac{\partial \bar{\Phi}}{\partial(\frac{1}{\bar{\rho}})} = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{2.119}$$

Using (2.119) in (2.118)

$$\dot{\bar{\Phi}} = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial({}^{(j)}\gamma_{ik})} \left({}^{(j)}\dot{\gamma}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial({}^{\bar{\Theta}}\bar{D}_{ik})} \left({}^{\bar{\Theta}}\dot{\bar{D}}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} \tag{2.120}$$

Substituting $\dot{\bar{\Phi}}$ from (2.120) in the entropy inequality (2.111)

$$\bar{\rho}\left(\bar{\rho}\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik}\right)} \left({}^{(j)}\dot{\gamma}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial({}^{\bar{\Theta}}\bar{D}_{ik})} \left({}^{\bar{\Theta}}\dot{\bar{D}}_{ik}\right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} + \bar{\eta}\dot{\bar{\theta}}\right)$$
$$+ \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_{s}^{(0)}\bar{\sigma}_{ik}{}^{(1)}\gamma_{ik} - {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) \leq 0 \tag{2.121}$$

Regrouping terms in (2.121)

$$\left(\bar{\rho}^2\frac{\partial \bar{\Phi}}{\partial \bar{\rho}}\delta_{ik} - {}_{s}^{(0)}\bar{\sigma}_{ik}\right){}^{(1)}\gamma_{ik} + \bar{\rho}\sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik}\right)} \left({}^{(j)}\dot{\gamma}_{ik}\right) + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial({}^{\bar{\Theta}}\bar{D}_{ik})} \left({}^{\bar{\Theta}}\dot{\bar{D}}_{ik}\right)$$
$$+ \bar{\rho}\left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta}\right)\dot{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} \leq 0 \tag{2.122}$$

For (2.122) to hold for arbitrary but admissible $[{}^{(j)}\dot{\gamma}]$; $j = 1, 2, \dots, n$, $[{}^{\bar{\Theta}}\dot{\bar{D}}]$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$, the following must hold

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{g}_i} = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{g}_i} = 0 \tag{2.123}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma_{ik})} = 0 \implies \frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma_{ik})} = 0 \quad ; \quad j = 1, 2, \dots, n \tag{2.124}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{\Theta}\bar{D}_{ik})} = 0 \implies \frac{\partial\bar{\Phi}}{\partial({}^{\Theta}\bar{D}_{ik})} = 0 \tag{2.125}$$

$$\bar{\rho}\left(\frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta}\right) = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta} = 0 \tag{2.126}$$

$$\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_s^{(0)}\bar{\sigma}_{ik}\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) \leq 0 \tag{2.127}$$

Equations (2.123) – (2.127) are fundamental relations from the entropy inequality

Remarks

1. Equation (2.123) implies that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{g}}$.

2. Equation (2.124) implies that $\bar{\Phi}$ is not a function of $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$.

3. Equation (2.125) implies that $\bar{\Phi}$ is not a function of $[{}^{\Theta}D]$.

4. Based on (2.126), $\bar{\eta}$ is not a dependent variable in the constitutive theories as $\bar{\eta} = -\frac{\partial\bar{\Phi}}{\partial\bar{\theta}}$, hence $\bar{\eta}$ is deterministic from $\bar{\Phi}$.

5. The last inequality is essential in the form it is stated. For example the following (or any other separation of terms)

$$\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_s^{(0)}\bar{\sigma}_{ik} = 0 \quad \text{and} \quad \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) \leq 0 \tag{2.128}$$

   are inappropriate due to the fact that these imply that ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$ is not a function of $\bar{\boldsymbol{g}}$, $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ as $\bar{\Phi}$ is not a function of these. This is contrary to (2.115). We also note that (2.127) in this form is unable to provide us with further details regarding the derivation of the constitutive theories.

In view of these remarks the arguments of the dependent variables in the constitutive theories in (2.115) can be modified. We can use $\bar{\rho}$ instead of $\frac{1}{\bar{\rho}}$.

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\bar{\rho},\, 0,\, 0,\, \bar{\theta},\, 0\right) \\
{}_s^{(0)}\bar{\boldsymbol{\sigma}} &= {}_s^{(0)}\bar{\boldsymbol{\sigma}}\left(\bar{\rho},\, [{}^{(k)}\gamma];\, k = 1, 2, \dots, n,\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\, [{}^{(k)}\gamma];\, k = 1, 2, \dots, n,\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{2.129}$$

We note that there is no mechanism or conditions that permit eliminating $[{}^{(k)}\gamma];\ k=1,2,\dots,n$ and $[{}^{\Theta}D]$ from the argument list of ${}^{(0)}\bar{\boldsymbol{q}}$, hence we must keep them as in (2.129). Based on (2.126) and remark (4), $\bar{\eta}$ is no longer a dependent variable in the constitutive theories. With (2.128) and (2.129) we have further mechanisms to proceed with the derivation of the constitutive theories.

2.4.1 Decomposition of stress tensor ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$

In order to remedy the situation discussed in remark (5), we consider decomposition of symmetric Cauchy stress tensor into equilibrium Cauchy stress tensor ${}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ and deviatoric Cauchy stress tensor ${}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ i.e.

$$
{}_{s}^{(0)}\bar{\boldsymbol{\sigma}} = {}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) + {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) \tag{2.130}
$$

in which we consider the following

$$
\begin{aligned}
{}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})\left(\bar{\rho},\,0,\,0,\,\bar{\theta},\,0\right)\\
{}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})\left(\bar{\rho},\,[{}^{(j)}\gamma];\ j=1,2,\dots,n,\,\bar{\theta},\,\bar{\boldsymbol{g}}\right)\\
\text{and}\quad {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})\left(\bar{\rho},\,0,\,\bar{\theta},\,0\right) = 0
\end{aligned} \tag{2.131}
$$

That is ${}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ is not a function of $[{}^{(j)}\gamma];\ j=1,2,\dots,n$ and $\bar{\boldsymbol{g}}$ and ${}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ vanishes when $[{}^{(j)}\gamma];\ j=1,2,\dots,n$ and $\bar{\boldsymbol{g}}$ are zero. Substituting (2.130) into (2.127)

$$
\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_{e}({}_{s}^{(0)}\bar{\sigma}_{ik}) - {}_{d}({}_{s}^{(0)}\bar{\sigma}_{ik})\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) \leq 0 \tag{2.132}
$$

or

$$
\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_{e}({}_{s}^{(0)}\bar{\sigma}_{ik})\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_{d}({}_{s}^{(0)}\bar{\sigma}_{ik})({}^{(1)}\gamma_{ik}) - {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) \leq 0 \tag{2.133}
$$

2.4.1.1 Constitutive theory for equilibrium stress ${}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$: compressible internal polar thermofluids

Since $\bar{\Phi}$ is not a function of ${}^{(1)}\gamma_{ik}$ and $\bar{\boldsymbol{g}}$ and neither is ${}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ (due to (2.131)), then the constitutive theory for ${}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})$ must be derivable from

$$
\begin{aligned}
{}_{e}({}_{s}^{(0)}\bar{\sigma}_{ik}) &= \bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} = \bar{p}(\bar{\rho},\bar{\theta})\delta_{ik}\\
\left[{}_{e}({}_{s}^{(0)}\bar{\sigma})\right] &= \bar{p}(\bar{\rho},\bar{\theta})[I]
\end{aligned} \tag{2.134}
$$

in which

$$
\bar{p}(\bar{\rho},\bar{\theta}) = \bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{2.135}
$$

$\bar{p}(\bar{\rho},\bar{\theta})$ is called thermodynamic pressure for compressible internal polar thermofluids and is generally referred to as an equation of state [71] in which $\bar{p}$ is expressed as a function of $\bar{\rho}$ and $\bar{\theta}$ or $\bar{v} = \frac{1}{\bar{\rho}}$ and $\bar{\theta}$, where $\bar{v}$ is specific volume. If we assume the compressive pressure to be positive, then $\bar{p}(\bar{\rho}, \bar{\theta})$ in (2.134) can be replaced by $-\bar{p}(\bar{\rho}, \bar{\theta})$. Using (2.134), inequality (2.133) reduces to

$$-_d({}_s^{(0)}\bar{\sigma}_{ik})({}^{(1)}\gamma_{ik}) + \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) \leq 0 \tag{2.136}$$

Inequality (2.136) is satisfied if

$${}^s\psi_d = {}_s^{(0)}\bar{\sigma}_{ik}\left({}^{(1)}\gamma_{ik}\right) > 0 \quad ; \quad {}^m\psi_d = {}^{(0)}\bar{m}_{ik}\left({}^{\Theta}\bar{D}_{ik}\right) > 0 \tag{2.137}$$

and

$$\frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0 \tag{2.138}$$

Inequalities (2.137) imply that the rate of work due to ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$ i.e. ${}^s\psi_d$ and due to ${}^{(0)}\bar{\boldsymbol{m}}$ i.e. ${}^m\psi_d$ must be positive. In view of (2.134) we can write the following for compressible internal polar thermofluids.

$$\begin{gathered}
[{}_s^{(0)}\bar{\sigma}] = \left[{}_e({}_s^{(0)}\bar{\sigma})(\bar{\rho},\, \bar{\theta})\right] + \left[{}_d({}_s^{(0)}\bar{\sigma})\left(\bar{\rho},\, [{}^{(k)}\gamma];\, k = 1, 2, \dots, n,\, \bar{\theta},\, \bar{\boldsymbol{g}}\right)\right] \\
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\, [{}^{(k)}\gamma];\, k = 1, 2, \dots, n,\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
\bar{\Phi} = \bar{\Phi}\left(\bar{\rho},\, \bar{\theta}\right) \\
\left[{}_e({}_s^{(0)}\bar{\sigma})\right] = \bar{p}(\bar{\rho},\, \bar{\theta})[I] \quad ; \quad \bar{p}(\bar{\rho},\, \bar{\theta}) = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}}
\end{gathered} \tag{2.139}$$

Constitutive theories for ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (2.137) and (2.138).

2.4.1.2 Constitutive theory for equilibrium stress ${}_e({}_s^{(0)}\bar{\boldsymbol{\sigma}})$: incompressible matter

For incompressible matter density is constant, hence $\bar{\rho} = \rho_0$, thus for this case $\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$, hence the constitutive theory for the incompressible case must consider $|J| = 1$ as $\bar{\rho} = \rho_0$. We must incorporate the incompressibility condition in the entropy inequality. We recall that incompressibility condition is given by (2.106)

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \operatorname{tr}[\bar{D}] = \operatorname{tr}[{}^{(1)}\gamma] = {}^{(1)}\gamma_{ik}\delta_{ik} = 0 \tag{2.140}$$

The incompressibility condition must be enforced. Based on (2.140) we can add

$$\bar{p}(\bar{\theta})\,{}^{(1)}\gamma_{ik}\delta_{ik} = 0 \tag{2.141}$$

to (2.133). $\bar{p}(\bar{\theta})$ is arbitrary Lagrange multiplier.

$$
\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_e({}_s^{(0)}\bar{\sigma}_{ik})\right){}^{(1)}\gamma_{ik} + \bar{p}(\bar{\theta}){}^{(1)}\gamma_{ik}\delta_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d({}_s^{(0)}\bar{\sigma}_{ik})({}^{(1)}\gamma_{ik}) - {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) \leq 0 \tag{2.142}
$$

Using $\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = 0$ in (2.142) and regrouping terms.

$$
\left(\bar{p}(\bar{\theta})\delta_{ik} - {}_e({}_s^{(0)}\bar{\sigma}_{ik})\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d({}_s^{(0)}\bar{\sigma}_{ik})({}^{(1)}\gamma_{ik}) - {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) \leq 0 \tag{2.143}
$$

In the case of incompressible internal polar thermofluids ${}_e({}_s^{(0)}\bar{\boldsymbol{\sigma}})$ is a function of $\bar{\theta}$ only, hence from (2.143) we have

$$
{}_e({}_s^{(0)}\bar{\sigma}_{ik}) = \bar{p}(\bar{\theta})\delta_{ik} \quad \text{or} \quad \left[{}_e({}_s^{(0)}\bar{\boldsymbol{\sigma}})\right] = \bar{p}(\bar{\theta})[I] \tag{2.144}
$$

$\bar{p}(\bar{\theta})$ is called mechanical pressure. Since $\bar{p}(\bar{\theta})$ is an arbitrary Lagrange multiplier, $\bar{p}(\bar{\theta})$ is not deterministic from the deformation field. In view of (2.144), (2.143) reduces to

$$
-{}_d({}_s^{(0)}\bar{\sigma}_{ik})\left({}^{(1)}\gamma_{ik}\right) - {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} \leq 0 \tag{2.145}
$$

Inequality (2.145) will hold if

$$
{}^s\psi_d = {}_d({}_s^{(0)}\bar{\sigma}_{ik})\left({}^{(1)}\gamma_{ik}\right) > 0; \quad {}^m\psi_d = {}^{(0)}\bar{m}_{ik}\left({}^{\bar{\Theta}}\bar{D}_{ik}\right) > 0 \tag{2.146}
$$

and

$$
\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} \leq 0 \tag{2.147}
$$

Conditions (2.146) and (2.147) are the same for the compressible case i.e. the rate of work due to ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$ and ${}^{(0)}\bar{\boldsymbol{m}}$ must be positive and the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (2.147). In view of (2.144) we can write the following for incompressible internal polar thermofluids.

$$
\begin{aligned}
[{}_s^{(0)}\bar{\sigma}] &= \left[{}_e({}_s^{(0)}\bar{\sigma})(\bar{\theta})\right] + \left[{}_d({}_s^{(0)}\bar{\sigma})\left(\bar{\rho}, [{}^{(j)}\gamma];\, j = 1, 2, \ldots, n,\, \bar{\theta},\, \bar{\boldsymbol{g}}\right)\right] \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho}, [{}^{(j)}\gamma];\, j = 1, 2, \ldots, n,\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left([{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
\bar{\Phi} &= \bar{\Phi}\left(\bar{\theta}\right) \\
\left[{}_e({}_s^{(0)}\bar{\sigma})\right] &= \bar{p}(\bar{\theta})[I]; \quad \frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = 0
\end{aligned} \tag{2.148}
$$

Constitutive theories for ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (2.146) and (2.147).

Remarks

1. Conditions resulting from the entropy inequality require decomposition of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into equilibrium and deviatoric stresses ${}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ and ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ (2.130) to proceed further.

2. Use of stress decomposition (2.130) in the conditions resulting from the entropy inequality permits determination of the constitutive theory for the equilibrium stress tensor for compressible as well as incompressible internal polar thermofluids in terms of thermodynamic pressure and mechanical pressure respectively.

3. The inequalities (2.136) or (2.145) require the rate of work due to ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ and ${}^{(0)}\bar{\boldsymbol{m}}$ be positive but provide no mechanisms for deriving constitutive theories for ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ and ${}^{(0)}\bar{\boldsymbol{m}}$.

4. The inequality (2.138) or (2.147) can be used (shown later) to derive a simple constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ (Fourier heat conduction law), but better constitutive theories are possible for ${}^{(0)}\bar{\boldsymbol{q}}$ (shown in subsequent sections).

5. The equilibrium stress ${}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ is independent of the basis for compressible as well as incompressible polar thermofluids due to the fact that $[I]$ is basis independent. This implies that

$$
\begin{aligned}
&\left[{}_{e}\big({}_{s}\bar{\sigma}^{(0)}\big)\right] = \left[{}_{e}\big({}_{s}\bar{\sigma}_{(0)}\big)\right] = \left[{}_{e}\big({}^{(0)}_{s}\bar{\sigma}^{J}\big)\right] = \bar{p}(\bar{\rho},\bar{\theta})[I]; \quad \text{Compressible matter} \\
\text{and} \quad &\left[{}_{e}\big({}_{s}\bar{\sigma}^{(0)}\big)\right] = \left[{}_{e}\big({}_{s}\bar{\sigma}_{(0)}\big)\right] = \left[{}_{e}\big({}^{(0)}_{s}\bar{\sigma}^{J}\big)\right] = \bar{p}(\bar{\theta})[I]; \quad \text{Incompressible matter}
\end{aligned}
\tag{2.149}
$$

2.4.2 Final choice of dependent variables in the constitutive theories and their argument tensors

The final choice of the dependent variables in the constitutive theories and their argument tensors for compressible and incompressible internal polar thermofluids are given by (2.139) and (2.148) and are summarized here for convenience.

2.4.2.1 Compressible internal polar thermofluids

In this case the dependent variables in the constitutive theories and their argument tensors are given by (2.139)

$$
\begin{gathered}
[{}^{(0)}_{s}\bar{\sigma}] = \left[{}_{e}({}^{(0)}_{s}\bar{\sigma})(\bar{\rho},\,\bar{\theta})\right] + \left[{}_{d}({}^{(0)}_{s}\bar{\sigma})\left(\bar{\rho},\,[{}^{(k)}\gamma];\,k=1,2,\ldots,n,\,\bar{\theta},\,\bar{\boldsymbol{g}}\right)\right] \\
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\,[{}^{(k)}\gamma];\,k=1,2,\ldots,n,\,[{}^{\Theta}D],\,\bar{\theta},\,\bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\,[{}^{\Theta}D],\,\bar{\theta},\,\bar{\boldsymbol{g}}\right) \\
\bar{\Phi} = \bar{\Phi}\left(\bar{\rho},\,\bar{\theta}\right) \\
\left[{}_{e}({}^{(0)}_{s}\bar{\sigma})\right] = \bar{p}(\bar{\rho},\,\bar{\theta})[I]; \quad \bar{p}(\bar{\rho},\,\bar{\theta}) = \bar{\rho}^{2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}
\end{gathered}
\tag{2.150}
$$

Constitutive theories for ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ and ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (2.137) and (2.138). Thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})$ is defined by an equation of state [71].

2.4.2.2 Incompressible internal polar thermofluids

The dependent variables in the constitutive theories and their argument tensors for incompressible internal polar thermofluids are given by (2.148).

$$
\begin{gathered}
[{}_s^{(0)}\bar{\sigma}] = \left[{}_e({}_s^{(0)}\bar{\sigma})(\bar{\theta})\right] + \left[{}_d({}_s^{(0)}\bar{\sigma})\left([{}^{(j)}\gamma];\, j=1,2,\dots,n,\, \bar{\theta},\, \bar{\boldsymbol{g}}\right)\right] \\
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left([{}^{(j)}\gamma];\, j=1,2,\dots,n,\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}\left([{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \\
\bar{\Phi} = \bar{\Phi}\left(\bar{\theta}\right) \\
\left[{}_e({}_s^{(0)}\bar{\sigma})\right] = \bar{p}(\bar{\theta})[I]; \quad \frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = 0
\end{gathered}
\tag{2.151}
$$

Constitutive theories for ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$ and ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (2.146) and (2.147). The mechanical pressure $\bar{p}(\bar{\theta})$ is not deterministic from deformation as it is an arbitrary Lagrange multiplier.

Remarks

We note that ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{q}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, $[{}^{(j)}\gamma]$; $j=1,2,\dots,n$ are basis dependent, hence must be chosen as $({}_d({}_s\bar{\boldsymbol{\sigma}}^{(0)}), \bar{\boldsymbol{q}}^{(0)}, \bar{\boldsymbol{m}}^{(0)}, [\gamma_{(j)}];\, j=1,2,\dots,n)$ and $({}_d({}_s\bar{\boldsymbol{\sigma}}_{(0)}), \bar{\boldsymbol{q}}_{(0)}, \bar{\boldsymbol{m}}_{(0)}, [\gamma^{(j)}];\, j=1,2,\dots,n)$ in contravariant and covariant bases and $({}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}}^J), {}^{(0)}\bar{\boldsymbol{q}}^J, {}^{(0)}\bar{\boldsymbol{m}}^J, [{}^{(j)}\gamma^J];\, j=1,2,\dots,n)$ when using Jaumann rates to obtain specific forms of the constitutive theories in the desired basis.

2.5 Constitutive theories for ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$: compressible matter

In the following we make some remarks that are helpful in understanding the approach used for deriving constitutive theories for ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$.

1. $[{}^{(j)}\gamma]$; $j=1,2,\dots,n$ are fundamental symmetric kinematic tensors of rank two. $[{}^{\Theta}D]$ is also a symmetric tensor of rank two. $\bar{\boldsymbol{g}}$ is a tensor of rank one and $\bar{\rho}$, $\bar{\theta}$ are tensors of rank zero.

2. $[{}^{(j)}\gamma]$; $j=1,2,\dots,n$, $[{}^{\Theta}D]$ and $\bar{\boldsymbol{g}}$ have their own generators and invariants but there also exist combined generators and invariants between them.

3. In the case of homogeneous isotropic compressible matter, the equilibrium stress is completely deterministic from the entropy inequality once we define Helmholtz free energy density in terms of invariants of the chosen strain measure. This yields thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})$. In the case of incompressible matter, the equilibrium stress is also derived from the entropy inequality in conjunction with the incompressibility constraint, however the equilibrium stress is not a function of Helmholtz free energy density and thus it is not deterministic from the deformation field [71]. Furthermore, the second law of thermodynamics only restricts the rate of work due to deviatoric stress and Cauchy moment tensors to be positive but provides no mechanisms for determining the constitutive theory for deviatoric stress ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$ and moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$.

4. The theory of generators and invariants [71, 76–92] provides a continuum mechanics foundation to derive constitutive equations for the deviatoric Cauchy stress tensor, Cauchy mo-

ment tensor, and heat vector. In this approach we determine the combined generators of the argument tensors of the dependent variable in the constitutive theory that form an integrity or minimal basis. The dependent variable in the constitutive theory is expressed as a linear combination of the combined generators of its argument tensors. In the case of the deviatoric Cauchy stress tensor $\left[{}_d({}^{(0)}_{s}\bar{\sigma})\right]$, a symmetric tensor of rank two, its argument tensors are $[{}^{(j)}\gamma];\ j=1,2,\dots,n$, symmetric tensors of rank two, $\bar{\boldsymbol{g}}$, a tensor of rank one, and $\bar{\rho}$, $\bar{\theta}$, tensors of rank zero. Thus, for $\left[{}_d({}^{(0)}_{s}\bar{\sigma})\right]$ we need combined generators of the tensors $[{}^{(j)}\gamma];\ j=1,2,\dots,n$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two. Then we express $\left[{}_d({}^{(0)}_{s}\bar{\sigma})\right]$ as a linear combination of these generators.

5. In the case of the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$, a symmetric tensor of rank two, its arguments are $[{}^{\Theta}D]$, a symmetric tensor of rank two and $\bar{\boldsymbol{g}}$, $\bar{\rho}$, $\bar{\theta}$ that are tensors of rank one, zero, and zero respectively. Thus, for ${}^{(0)}\bar{\boldsymbol{m}}$ we need combined generators of $[{}^{\Theta}D]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two. Then we can express ${}^{(0)}\bar{\boldsymbol{m}}$ as a linear combination of these generators.

6. In the case of ${}^{(0)}\bar{\boldsymbol{q}}$, a tensor of rank one, with its arguments $[{}^{(j)}\gamma];\ j=1,2,\dots,n$, $[{}^{\Theta}D]$, $\bar{\boldsymbol{g}}$, and $\bar{\theta}$ we need combined generators of $[{}^{(j)}\gamma];\ j=1,2,\dots,n$, $[{}^{\Theta}D]$, and $\bar{\boldsymbol{g}}$ that are tensors of rank one. Then we can express ${}^{(0)}\bar{\boldsymbol{q}}$ as a linear combination of these generators.

7. In the linear combination of the generators for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$, the coefficients in the linear combinations are functions of the combined invariants of the corresponding argument tensors. The material coefficients are determined by using Taylor series expansion of these coefficients about a known configuration.

8. In the following section we first consider derivations of the constitutive theories for compressible polar thermofluids. These constitutive theories are then modified for incompressible polar thermofluids and are presented in the subsequent sections.

2.5.1 Rate constitutive theories of up to order n for deviatoric symmetric Cauchy stress tensor ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$: compressible

We consider (from (2.150))

$$
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})\left(\bar{\rho}, [{}^{(j)}\gamma];\ j=1,2,\dots,n,\ \bar{\theta},\ \bar{\boldsymbol{g}}\right) \tag{2.152}
$$

Let $[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i];\ i=1,2,\dots,N$ be the combined generators of the argument tensors $[{}^{(j)}\gamma];\ j=1,2,\dots,n$, and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two [71] and ${}^{\sigma}\underset{\sim}{I}^j;\ j=1,2,\dots,M$ be the combined invariants of the same argument tensors [71]. Then, we can express $\left[{}_d({}^{(0)}_{s}\bar{\sigma})\right]$ as a linear combination of $[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i];\ i=1,2,\dots,N$ and identity tensor $[I]$.

$$
\left[{}_d({}^{(0)}_{s}\bar{\sigma})\right] = {}^{\sigma}\underset{\sim}{\alpha}^0 + \sum_{i=1}^{N} {}^{\sigma}\underset{\sim}{\alpha}^i [{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i] \tag{2.153}
$$

The coefficients ${}^{\sigma}\underset{\sim}{\alpha}^{0}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\dots,M$ in the current configuration

$$ {}^{\sigma}\underset{\sim}{\alpha}^{k} = {}^{\sigma}\underset{\sim}{\alpha}^{k}\left(\rho,\ {}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\dots,M,\ \bar{\theta}\right) \tag{2.154} $$

To determine material coefficients from (2.154), we consider Taylor series expansion of each ${}^{\sigma}\underset{\sim}{\alpha}^{k};\ k=0,1,\dots,N$ in θ, ${}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\dots,M$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in $\bar{\theta}$ and the invariants (for simplicity).

$$ {}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right);\ i=0,1,\dots,N \tag{2.155} $$

We note that ${}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}}$, $\left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}};\ j=1,2,\dots,M$, $\left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial\bar{\theta}}\right|_{\underline{\Omega}};\ i=0,1,\dots,N$ are functions of $\bar{\rho}\big|_{\underline{\Omega}}$, $\bar{\theta}\big|_{\underline{\Omega}}$ and $({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}$ whereas ${}^{\sigma}\underset{\sim}{\alpha}^{i}$ are functions of the same quantities but in the current configuration (2.154). When (2.155) is substituted in (2.153), we obtain the final expression for the most general rate constitutive theory of up to order n for ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ for compressible polar thermofluids. The final expression defines the material coefficients in the known configuration $\underline{\Omega}$. Details are given in the following. First, substitute (2.155) in (2.153),

$$ \begin{aligned} \left[{}_{d}({}^{(0)}_{s}\bar{\sigma})\right] = & \left({}^{\sigma}\underset{\sim}{\alpha}^{0}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{0})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{0})}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\right)[I] \\ & + \sum_{i=1}^{N} \left({}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\right)\left[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right] \end{aligned} \tag{2.156} $$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (2.156) that are defined in the current configuration and also grouping those terms that are completely defined in the known configuration $\underline{\Omega}$. Let us define

$$
\begin{aligned}
{}^{0}\bar{\sigma}\big|_{\underline{\Omega}} &= \left({}^{\sigma}\underset{\sim}{\alpha}^{0}\big|_{\underline{\Omega}} - \sum_{j=1}^{M} \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{0})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) \\
{}^{\sigma}\underline{a}_{j} &= \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{0})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \; ; \quad j = 1, 2, \dots, M \\
{}^{\sigma}\underline{b}_{i} &= {}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}} - \sum_{j=1}^{M} \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{0})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \\
{}^{\sigma}\underline{c}_{ij} &= \left.\frac{\partial({}^{\sigma}\underset{\sim}{\alpha}^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\right|_{\underline{\Omega}} \; ; \quad \begin{array}{l} i = 1, 2, \dots, N \\ j = 1, 2, \dots, M \end{array} \\
\underline{\alpha}_{\text{tm}} &= - \left.\frac{\partial {}^{\sigma}\underset{\sim}{\alpha}^{0}}{\partial \theta}\right|_{\underline{\Omega}} \\
{}^{\sigma}\underline{d}_{i} &= - \left.\frac{\partial {}^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial \bar{\theta}}\right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, N
\end{aligned}
\tag{2.157}
$$

Using (2.157), we can write (2.156) as follows

$$
\begin{aligned}
\left[{}_{d}({}_{s}^{(0)}\bar{\sigma})\right] = {}^{0}\bar{\sigma}\big|_{\underline{\Omega}} [I] + \sum_{j=1}^{M} {}^{\sigma}\underline{a}_{j} {}^{\sigma}\underset{\sim}{I}^{j} [I] - \underline{\alpha}_{\text{tm}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right) [I] + \sum_{i=1}^{N} {}^{\sigma}\underline{b}_{i} \left[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right] \\
+ \sum_{i=1}^{N} \sum_{j=1}^{M} {}^{\sigma}\underline{c}_{ij} {}^{\sigma}\underset{\sim}{I}^{j} \left[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right] - \sum_{i=1}^{N} {}^{\sigma}\underline{d}_{i} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right) \left[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right]
\end{aligned}
\tag{2.158}
$$

${}^{\sigma}\underline{a}_{j}$, ${}^{\sigma}\underline{b}_{i}$, ${}^{\sigma}\underline{c}_{ij}$, ${}^{\sigma}\underline{d}_{i}$ and $\underline{\alpha}_{\text{tm}}$ are material coefficients defined in known configurations $\underline{\Omega}$. This constitutive theory requires $(M + N + (M)(N) + N + 1)$ material coefficients. The material coefficients defined in (2.158) are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and ${}^{\sigma}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, M$. This constitutive theory is based on integrity, hence it is complete.

2.5.2 Rate constitutive theories of up to order n for heat vector ${}^{(0)}\bar{\boldsymbol{q}}$: compressible

Consider (from (2.150))

$$
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}} \left(\bar{\rho},\, [{}^{(k)}\gamma];\, k = 1, 2, \dots, n,\, [{}^{\Theta}D],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right) \tag{2.159}
$$

Let $\left\{{}^{q}\underset{\sim}{\boldsymbol{G}}^{i}\right\};\ i = 1, 2, \dots, \tilde{N}$ be the combined generators of the argument tensors $[{}^{(j)}\gamma];\ j = 1, 2, \dots, n,\ [{}^{\Theta}D]$, and $\bar{\boldsymbol{g}}$ that are tensors of rank one. Let ${}^{q}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, \tilde{M}$ be the combined invariants of the same argument tensors. Then, we can express $\left\{{}^{(0)}\bar{q}\right\}$ as a linear combination of

$\left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\}$; $i = 1, 2, \dots, \tilde{N}$.

$$\left\{ {}^{(0)}\bar{q} \right\} = -\sum_{i=1}^{\tilde{N}} {}^q\underset{\sim}{\alpha}^i \left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\} \tag{2.160}$$

The absence of unit vector in (2.160) is due to the fact that a uniform temperature field does not contribute to $\left\{ {}^{(0)}\bar{q} \right\}$. The negative sign in (2.160) is because a positive $\left\{ {}^{(0)}\bar{q} \right\}$ in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter. The coefficients ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^q\underset{\sim}{I}^j$; $j = 1, 2, \dots, \tilde{M}$ in the current configuration. To determine the material coefficients from ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ (in the current configuration) in (2.160), we consider Taylor series expansion of each ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ about a known configuration $\underline{\Omega}$ in $\bar{\theta}$ and ${}^q\underset{\sim}{I}^j$; $j = 1, 2, \dots, \tilde{M}$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial({}^q\underset{\sim}{I}^j)} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}} \right|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}} \right) ;\ i = 1, 2, \dots, \tilde{N} \tag{2.161}
$$

${}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}}$, $\left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial({}^q\underset{\sim}{I}^j)} \right|_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$ and $\left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}} \right|_{\underline{\Omega}}$; $i = 1, 2, \dots, \tilde{N}$ are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$, and $\left({}^q\underset{\sim}{I}^j \right)_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$ whereas ${}^q\underset{\sim}{\alpha}^i$ are functions of the same quantities in the current configuration. When (2.161) is substituted in (2.160) we obtain the most general n^{th} order rate constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$. Details are presented in the following.

$$
\left\{ {}^{(0)}\bar{q} \right\} = -\sum_{i=1}^{\tilde{N}} \left({}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial({}^q\underset{\sim}{I}^j)} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}} \right|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}} \right) \right) \left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\} \tag{2.162}
$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (2.162) that are defined in the current configuration i.e. coefficients of $\left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\}$, ${}^q\underset{\sim}{I}^j \left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\}$ and $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}) \left\{ {}^q\underset{\sim}{\boldsymbol{G}}^i \right\}$.

$$
\begin{aligned}
{}^q\underline{b}_i &= {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{\tilde{M}} \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial({}^q\underset{\sim}{I}^j)} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j \right)_{\underline{\Omega}} \\
{}^q\underline{c}_{ij} &= \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial({}^q\underset{\sim}{I}^j)} \right|_{\underline{\Omega}} \\
{}^q\underline{d}_i &= \left. \frac{\partial({}^q\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}} \right|_{\underline{\Omega}}
\end{aligned} \tag{2.163}
$$

for $i = 1, 2, \dots, \tilde{N}$ and $j = 1, 2, \dots, \tilde{M}$.

Using (2.163) in (2.162), we can write (2.162) as

$$\left\{{}^{(0)}\bar{q}\right\} = -\sum_{i=1}^{\tilde{N}} {}^{q}\underline{b}_i \left\{{}^{q}\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}}\sum_{j=1}^{\tilde{M}} {}^{q}\underline{c}_{ij}\, {}^{q}\underset{\sim}{I}^j \left\{{}^{q}\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}} {}^{q}\underline{d}_i \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right) \left\{{}^{q}\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{2.164}$$

${}^{q}\underline{b}_i$, ${}^{q}\underline{c}_{ij}$, and ${}^{q}\underline{d}_i$ are material coefficients defined in known configuration $\underline{\Omega}$. This constitutive theory defined by (2.164) requires $(\tilde{N}+\tilde{N}\tilde{M}+\tilde{N})$ material coefficients. The material coefficients are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and $({}^{q}\underset{\sim}{I}^j)_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$. This theory is based on integrity, hence is complete.

2.5.3 Constitutive theory for Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$: compressible

Consider the following (from (2.150))

$$[{}^{(0)}\bar{m}] = \left[{}^{(0)}\bar{m}\left(\bar{\rho},\, [{}^{\Theta}D],\, \bar{\theta},\, \{\bar{g}\}\right)\right] \tag{2.165}$$

The combined generators of the argument tensors $[{}^{\Theta}D]$ and $\{\bar{g}\}$ that are symmetric tensors of rank two are given by [71]

$$\begin{aligned}
&\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^1\right] = [{}^{\Theta}D] \quad , \quad \left[{}^{m}\underset{\sim}{\boldsymbol{G}}^2\right] = [{}^{\Theta}D]^2 \quad , \quad \left[{}^{m}\underset{\sim}{\boldsymbol{G}}^3\right] = \{g\}\{g\}^T \\
&\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^4\right] = \{g\}\left\{[{}^{\Theta}D]\{g\}\right\}^T + \left\{[{}^{\Theta}D]\{g\}\right\}\{g\}^T \\
&\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^5\right] = \{g\}\left\{[{}^{\Theta}D]^2\{g\}\right\}^T + \left\{[{}^{\Theta}D]^2\{g\}\right\}\{g\}^T
\end{aligned} \tag{2.166}$$

The combined invariants of the tensors $[{}^{\Theta}D]$ and $\{g\}$ are given by [71].

$$\begin{aligned}
&{}^{m}\underset{\sim}{I}^1 = \mathrm{tr}[{}^{\Theta}D] \quad , \quad {}^{m}\underset{\sim}{I}^2 = \mathrm{tr}\left([{}^{\Theta}D]^2\right) \quad , \quad {}^{m}\underset{\sim}{I}^3 = \mathrm{tr}\left([{}^{\Theta}D]^3\right) \\
&{}^{m}\underset{\sim}{I}^4 = \{g\}^T\{g\} \quad , \quad {}^{m}\underset{\sim}{I}^5 = \{g\}^T\left\{[{}^{\Theta}D]\{g\}\right\} \\
&{}^{m}\underset{\sim}{I}^6 = \{g\}^T\left\{[{}^{\Theta}D]^2\{g\}\right\}
\end{aligned} \tag{2.167}$$

We can express $[{}^{(0)}\bar{m}]$ as a linear combination of $\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^i\right]$; $i = 1, 2, \dots, 5$ and identity tensor $[I]$.

$$[{}^{(0)}\bar{m}] = {}^{m}\underset{\sim}{\alpha}^0[I] + \sum_{i=1}^{\underset{\sim}{N}} {}^{m}\underset{\sim}{\alpha}^i\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^i\right]; \quad \underset{\sim}{N} = 5 \tag{2.168}$$

The coefficients ${}^{m}\underset{\sim}{\alpha}^i$; $i = 0, 1, \dots, \underset{\sim}{N}$ are functions of $\bar{\rho}, \bar{\theta}$ and ${}^{m}\underset{\sim}{I}^j$; $j = 1, 2, \dots, \underset{\sim}{M}$ $(\underset{\sim}{M} = 6)$ in the current configuration i.e.

$${}^{m}\underset{\sim}{\alpha}^i = {}^{m}\underset{\sim}{\alpha}^i\left(\bar{\rho},\, \bar{\theta},\, {}^{m}\underset{\sim}{I}^j;\ j = 1, 2, \dots, \underset{\sim}{M}\right); \quad i = 0, 1, \dots, \underset{\sim}{N} \tag{2.169}$$

To determine material coefficients from (2.169), we consider Taylor series expansion of each ${}^{m}\underset{\sim}{\alpha}^i$; $i = 0, 1, \dots, \underset{\sim}{N}$ in $\bar{\theta}$, ${}^{m}\underset{\sim}{I}^j$; $j = 1, 2, \dots, \underset{\sim}{M}$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in $\bar{\theta}$ and the invariants (for simplicity). Using $\underset{\sim}{N} = 5$ and $\underset{\sim}{M} = 6$, we can write

$$
{}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{\underset{\sim}{M}} \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - \left({}^m\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right); \quad i = 0, 1, \dots, \underset{\sim}{N} \tag{2.170}
$$

We note that ${}^m\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}}$, $\left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}}$; $j = 1, 2, \dots, \underset{\sim}{M}$, $\left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}}$; $i = 0, 1, \dots, \underset{\sim}{N}$ are functions of $\bar{\rho}\big|_{\underline{\Omega}}$, $\bar{\theta}\big|_{\underline{\Omega}}$, and $\left({}^m\underset{\sim}{I}^j\right)_{\underline{\Omega}}$; $j = 1, 2, \dots, \underset{\sim}{M}$. We substitute (2.170) in (2.168) to obtain the most general constitutive theory for $[{}^{(0)}\bar{m}]$. Details are given in the following.

$$
\begin{aligned}
[{}^{(0)}\bar{m}] = {} & \left({}^m\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} + \sum_{j=1}^{\underset{\sim}{M}} \left.\frac{\partial({}^m\underset{\sim}{\alpha}^0)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - ({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial({}^m\underset{\sim}{\alpha}^0)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \right) [I] \\
& + \left({}^m\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{\underset{\sim}{M}} \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - ({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right) \right) \left[{}^m\underset{\sim}{\boldsymbol{G}}^i\right]
\end{aligned} \tag{2.171}
$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (2.171) that are defined in the current configuration and also grouping those terms that are completely defined in the known configuration $\underline{\Omega}$.

Let us define

$$
\begin{aligned}
{}^0\bar{m}\big|_{\underline{\Omega}} &= {}^m\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} - \sum_{j=1}^{\underset{\sim}{M}} \left.\frac{\partial({}^m\underset{\sim}{\alpha}^0)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^m\underline{a}_j &= \left.\frac{\partial({}^m\underset{\sim}{\alpha}^0)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}}; \quad j = 1, 2, \dots, \underset{\sim}{M} \\
{}^m\underline{b}_i &= {}^m\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{\underset{\sim}{M}} \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^m\underline{c}_{ij} &= \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}}; \quad \begin{array}{l} i = 1, 2, \dots, \underset{\sim}{N} \\ j = 1, 2, \dots, \underset{\sim}{M} \end{array} \\
{}^m\underline{\alpha}_{\mathrm{tm}} &= \left.\frac{\partial({}^m\underset{\sim}{\alpha}^0)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \\
{}^m\underline{a}_j &= \left.\frac{\partial({}^m\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}}; \quad i = 1, 2, \dots, \underset{\sim}{N}
\end{aligned} \tag{2.172}
$$

Using (2.172) in (2.171) we can rewrite (2.172) as follows

$$
\begin{aligned}
[{}^{(0)}\bar{m}] = {}^{0}\bar{m}\big|_{\underline{\Omega}}\,[I] &+ \sum_{j=1}^{\underset{\sim}{M}} {}^{m}\underline{a}_j\, {}^{m}\underset{\sim}{I}^{j}\,[I] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] + \sum_{i=1}^{\underset{\sim}{N}} {}^{m}\underline{b}_i\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right] \\
&+ \sum_{i=1}^{\underset{\sim}{N}} \sum_{j=1}^{\underset{\sim}{M}} {}^{m}\underline{c}_{ij}\, {}^{m}\underset{\sim}{I}^{j}\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right] - \sum_{i=1}^{\underset{\sim}{N}} {}^{m}\underline{d}_i\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right]
\end{aligned}
\tag{2.173}
$$

${}^{m}\underline{a}_j$, ${}^{m}\underline{b}_i$, ${}^{m}\underline{c}_{ij}$, ${}^{m}\underline{d}_i$ and ${}^{m}\underline{\alpha}_{\mathrm{tm}}$ are material coefficients defined in known configurations $\underline{\Omega}$. This constitutive theory requires $(\underset{\sim}{M} + \underset{\sim}{N} + (\underset{\sim}{N})(\underset{\sim}{M}) + \underset{\sim}{N} + 1)$ material coefficients (forty-seven). The material coefficients defined in (2.172) are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and $({}^{m}\underset{\sim}{I}^{j})_{\underline{\Omega}}$; $j = 1, 2, \dots, \underset{\sim}{M}$. This constitutive theory is based on integrity, hence is complete.

Remarks

1. The constitutive theories presented in sections 2.5.1–2.5.3 utilize ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{q}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ as dependent variables with $\bar{\rho}$, $[{}^{(k)}\gamma]$; $k = 1, 2, \dots, n$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors of ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ and $\bar{\rho}$, $[{}^{(k)}\gamma]$; $k = 1, 2, \dots, n$, $[{}^{\Theta}D]$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors of ${}^{(0)}\bar{\boldsymbol{q}}$, whereas $\bar{\rho}$, $[{}^{\Theta}D]$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ are considered as argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$. Hence, these derivations are independent of the basis.

2. By replacing ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$, ${}^{(0)}\bar{\boldsymbol{q}}$, ${}^{(0)}\bar{\boldsymbol{m}}$ and $[{}^{(k)}\gamma]$; $k = 1, 2, \dots, n$ with the appropriate corresponding measures in the chosen basis, we can obtain the rate constitutive theories for the deviatoric symmetric Cauchy stress tensor, heat vector, and Cauchy moment tensor in the desired basis. More specifically we use the following measures.

$$
\begin{aligned}
\text{Contravariant basis:}\quad & {}_{d}({}_{s}\bar{\boldsymbol{\sigma}}^{(0)}),\ \bar{\boldsymbol{q}}^{(0)},\ [\gamma_{(j)}];\ j = 1, 2, \dots, n,\ \bar{\boldsymbol{m}}^{(0)} \\
\text{Covariant basis:}\quad & {}_{d}({}_{s}\bar{\boldsymbol{\sigma}}_{(0)}),\ \bar{\boldsymbol{q}}_{(0)},\ [\gamma^{(j)}];\ j = 1, 2, \dots, n,\ \bar{\boldsymbol{m}}_{(0)} \\
\text{Jaumann:}\quad & {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}^{J}),\ {}^{(0)}\bar{\boldsymbol{q}}^{J},\ [{}^{(j)}\gamma^{J}];\ j = 1, 2, \dots, n,\ {}^{(0)}\bar{\boldsymbol{m}}^{J}
\end{aligned}
\tag{2.174}
$$

3. The configuration $\underline{\Omega}$ can be chosen to be the reference configuration (undeformed configuration before the commencement of the evolution) in which case the material coefficients will be independent of the deformation. If we choose $\underline{\Omega}$ to be a known deformation configuration, then the deformation dependent material coefficients are possible in the constitutive theories. Dependence of the material coefficients on the invariants of the argument tensor in a known configuration $\underline{\Omega}$ permits complex description of material coefficients.

4. An important point to note is that the material coefficients in the final forms of the constitutive theories are defined in a known configuration $\underline{\Omega}$, whereas the constitutive equations hold in the current configuration for which the deformation field is yet to be determined. This of course is a consequence of the Taylor series expansion of the coefficients in the linear combination (using combined generators) about a known configuration. In the currently used constitutive models in the published works [93] for variable material coefficients, the coefficients are expressed as

a function of the unknown deformation field in the current configuration. This is obviously not supported by the derivations presented in sections 2.5.1–2.5.3.

5. Using the derivations presented in sections 2.5.1–2.5.3 rate constitutive theories of various orders in desired basis can be derived by choosing a value of n, the order of the rate theory. As the order of the rate theory increases, the number of material constants increase significantly. Thus, the higher order rate theories necessitate elaborate experiments to calibrate them.

6. In the following we consider rate theories of order one ($n = 1$) and their further simplifications to present rather simple theories that could be used as model problems.

2.5.4 Rate constitutive theories of order one ($n = 1$) for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$: compressible

This is the simples possible constitutive theory for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ in which there is interaction between $[{}^{(1)}\gamma]$ and $\{g\}$. We consider

$$\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = \left[{}_d({}^{(0)}_s\bar{\sigma})\left(\bar{\rho},\, [{}^{(1)}\gamma],\, \bar{\theta},\, \bar{\boldsymbol{g}}\right)\right] \tag{2.175}$$

In this case the combined generators of $[{}^{(1)}\gamma]$ and $\{g\}$ that are symmetric tensors of rank two are ($N = 5$)

$$\begin{aligned}
&[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1] = [{}^{(1)}\gamma] \quad , \quad [{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^2] = [{}^{(1)}\gamma]^2 \quad , \quad [{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^3] = \{g\}\{g\}^T \\
&[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^4] = \{g\}\left\{[{}^{(1)}\gamma]\{g\}\right\}^T + \left\{[{}^{(1)}\gamma]\{g\}\right\}\{g\}^T \\
&[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^5] = \{g\}\left\{[{}^{(1)}\gamma]^2\{g\}\right\}^T + \left\{[{}^{(1)}\gamma]^2\{g\}\right\}\{g\}^T
\end{aligned} \tag{2.176}$$

and the combined invariants of $[{}^{(1)}\gamma]$, $\{g\}$ are ($M = 6$)

$$\begin{aligned}
&{}^{\sigma}\underset{\sim}{I}^1 = \operatorname{tr}[{}^{(1)}\gamma] \quad , \quad {}^{\sigma}\underset{\sim}{I}^2 = \operatorname{tr}\left([{}^{(1)}\gamma]^2\right) \quad , \quad {}^{\sigma}\underset{\sim}{I}^3 = \operatorname{tr}\left([{}^{(1)}\gamma]^3\right) \\
&{}^{\sigma}\underset{\sim}{I}^4 = \{g\}^T\{g\} \quad , \quad {}^{\sigma}\underset{\sim}{I}^5 = \{g\}^T\left\{[{}^{(1)}\gamma]\{g\}\right\} \\
&{}^{\sigma}\underset{\sim}{I}^6 = \{g\}^T\left\{[{}^{(1)}\gamma]^2\{g\}\right\}
\end{aligned} \tag{2.177}$$

Thus, we can write

$$\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}^{\sigma}\underset{\sim}{\alpha}^0 + \sum_{i=1}^{5} {}^{\sigma}\underset{\sim}{\alpha}^i\,[{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i] \tag{2.178}$$

Following the general derivations in section 2.5 for N generators and M invariants, for this specific case we can write

$$
\begin{aligned}
\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}&{}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + \sum_{j=1}^{6}{}^\sigma\underline{a}_j{}^\sigma\underset{\sim}{I}^j[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] + \sum_{i=1}^{5}{}^\sigma\underline{b}_i\left[{}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right] \\
&+ \sum_{i=1}^{5}\sum_{j=1}^{6}{}^\sigma\underline{c}_{ij}{}^\sigma\underset{\sim}{I}^j\left[{}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right] - \sum_{i=1}^{5}{}^\sigma\underline{d}_i\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right]
\end{aligned}
\tag{2.179}
$$

The definitions of material coefficients ${}^\sigma\underline{a}_j$, ${}^\sigma\underline{b}_i$, ${}^\sigma\underline{c}_{ij}$, ${}^\sigma\underline{d}_i$ and $\underline{\alpha}_{\text{tm}}$ as well as ${}^0\bar{\sigma}\big|_{\underline{\Omega}}$ remain the same as defined in (2.157). This constitutive theory requires 46 material coefficients, still too many to determine experimentally.

2.5.4.1 Simplified rate constitutive theory of order one ($n = 1$) for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$: compressible

Consider a constitutive theory in which ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ is not dependent on $\{g\}$ i.e.

$$
\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = \left[{}_d({}^{(0)}_s\bar{\sigma})\left(\bar{\rho},\, [{}^{(1)}\gamma],\, \bar{\theta}\right)\right]
\tag{2.180}
$$

In this case we have only two generators ($N = 2$) and three invariants ($M = 3$)

$$
\begin{aligned}
&\left[{}^\sigma\underset{\sim}{\boldsymbol{G}}^1\right] = [{}^{(1)}\gamma] \quad , \quad \left[{}^\sigma\underset{\sim}{\boldsymbol{G}}^2\right] = [{}^{(1)}\gamma]^2 \\
&{}^\sigma\underset{\sim}{I}^1 = \text{tr}[{}^{(1)}\gamma] \quad , \quad {}^\sigma\underset{\sim}{I}^2 = \text{tr}\left([{}^{(1)}\gamma]^2\right) \quad , \quad {}^\sigma\underset{\sim}{I}^3 = \text{tr}\left([{}^{(1)}\gamma]^3\right)
\end{aligned}
\tag{2.181}
$$

and we have the following constitutive theory (using (2.158) for $N = 2$ and $M = 3$)

$$
\begin{aligned}
\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}&{}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\text{tr}[{}^{(1)}\gamma][I] + {}^\sigma\underline{a}_2\text{tr}[{}^{(1)}\gamma]^2[I] + {}^\sigma\underline{a}_3\text{tr}[{}^{(1)}\gamma]^3[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] \\
&+ {}^\sigma\underline{b}_1[{}^{(1)}\gamma] + {}^\sigma\underline{b}_2[{}^{(1)}\gamma]^2 + {}^\sigma\underline{c}_{11}\left(\text{tr}[{}^{(1)}\gamma]\right)[{}^{(1)}\gamma] + {}^\sigma\underline{c}_{12}\left(\text{tr}[{}^{(1)}\gamma]^2\right)[{}^{(1)}\gamma] \\
&+ {}^\sigma\underline{c}_{13}\left(\text{tr}[{}^{(1)}\gamma]^3\right)[{}^{(1)}\gamma] + {}^\sigma\underline{c}_{21}\left(\text{tr}[{}^{(1)}\gamma]\right)[{}^{(1)}\gamma]^2 + {}^\sigma\underline{c}_{22}\left(\text{tr}[{}^{(1)}\gamma]^2\right)[{}^{(1)}\gamma]^2 \\
&+ {}^\sigma\underline{c}_{23}\left(\text{tr}[{}^{(1)}\gamma]^3\right)[{}^{(1)}\gamma]^2 + {}^\sigma\underline{d}_1\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[{}^{(1)}\gamma] + {}^\sigma\underline{d}_2\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[{}^{(1)}\gamma]^2
\end{aligned}
\tag{2.182}
$$

This constitutive theory requires 14 material coefficients and contains up to fifth degree terms in the components of $[{}^{(1)}\gamma]$.

2.5.4.2 Simplified rate constitutive theory of order one ($n = 1$) for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ that is quadratic in the components of $[{}^{(1)}\gamma]$: compressible

We can begin with (2.182) and neglect those terms on the right side of (2.182) that are degree higher than two in the components of $[{}^{(1)}\gamma]$.

$$\begin{aligned}\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}& {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\mathrm{tr}[{}^{(1)}\gamma][I] + {}^\sigma\underline{a}_2\mathrm{tr}[{}^{(1)}\gamma]^2[I] - \underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \\ & + {}^\sigma\underline{b}_1[{}^{(1)}\gamma] + {}^\sigma\underline{b}_2[{}^{(1)}\gamma]^2 + {}^\sigma\underline{c}_{11}\left(\mathrm{tr}[{}^{(1)}\gamma]\right)[{}^{(1)}\gamma] \\ & + {}^\sigma\underline{d}_1\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{(1)}\gamma] + {}^\sigma\underline{d}_2\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{(1)}\gamma]^2\end{aligned} \tag{2.183}$$

This constitutive theory requires 8 material coefficients.

If we further neglect the product terms in $\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)$ (last two terms on the right side of (2.183)) in (2.183) we obtain

$$\begin{aligned}\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}& {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\mathrm{tr}[{}^{(1)}\gamma][I] + {}^\sigma\underline{a}_2\mathrm{tr}[{}^{(1)}\gamma]^2[I] - \underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \\ & + {}^\sigma\underline{b}_1[{}^{(1)}\gamma] + {}^\sigma\underline{b}_2[{}^{(1)}\gamma]^2 + {}^\sigma\underline{c}_{11}\left(\mathrm{tr}[{}^{(1)}\gamma]\right)[{}^{(1)}\gamma]\end{aligned} \tag{2.184}$$

This constitutive theory requires only six material coefficients. The dependence of the material coefficients on the invariants in (2.184) can be modified based on the assumptions used here or can be modified based on the assumptions we used here or can be maintained as originally defined in (2.157).

2.5.4.3 Simplified rate constitutive theory of order one ($n = 1$) for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ that is linear in the components of $[{}^{(1)}\gamma]$: compressible

If we neglect quadratic terms in $[{}^{(1)}\gamma]$ in (2.184), then we obtain a constitutive theory for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ that is linear in $[{}^{(1)}\gamma]$.

$$\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\mathrm{tr}[{}^{(1)}\gamma][I] + {}^\sigma\underline{b}_1[{}^{(1)}\gamma] - \underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{2.185}$$

If we denote $\kappa_{\underline{\Omega}} = {}^\sigma\underline{a}_1$ and $2\eta_{\underline{\Omega}} = {}^\sigma\underline{b}_1$, then we can write (2.185) as

$$\left[{}_d({}^{(0)}_s\bar{\sigma})\right] = {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + 2\eta_{\underline{\Omega}}[{}^{(1)}\gamma] + \kappa_{\underline{\Omega}}\mathrm{tr}[{}^{(1)}\gamma][I] - \underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{2.186}$$

Material coefficients $\eta_{\underline{\Omega}}$ and $\kappa_{\underline{\Omega}}$ can be functions of $\bar{\rho}|_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $[{}^{(1)}\gamma]$ in the known configuration $\underline{\Omega}$. The constitutive theory (2.186) is the simplest possible constitutive theory for deviatoric symmetric Cauchy stress tensors.

2.5.5 Remarks on constitutive theories for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$: compressible

1. We note that the arguments of ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ are same as those of ${}^{(0)}_d\bar{\boldsymbol{\sigma}}$, deviatoric Cauchy stress tensor for nonpolar thermofluids [71, 94]. Thus the constitutive theories for ${}_d({}^{(0)}_s\bar{\boldsymbol{\sigma}})$ for polar thermofluids are the same as those for ${}^{(0)}_d\bar{\boldsymbol{\sigma}}$ for nonpolar thermofluids. The fundamental dif-

ference is that even though the constitutive theories are the same, they are for different stress measures. ${}^{(0)}_{d}\bar{\boldsymbol{\sigma}}$ is the deviatoric part of the total Cauchy stress tensor, whereas ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ is the deviatoric part of the symmetric part of the Cauchy stress tensor.

2. We make some specific remarks for the simplified rate theory of order one given by (2.186). When we compare (2.186) with the similar theory for ${}^{(0)}_{d}\bar{\boldsymbol{\sigma}}$, we note that η and κ are similar to first and second viscosities and $\underline{\alpha}_{\text{tm}}$ is thermal modulus. Since

$$[{}^{(1)}\gamma] = [\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}] \tag{2.187}$$

equation (2.186) implies that

$${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_{d}({}_{s}\bar{\boldsymbol{\sigma}}^{(0)}) = {}_{d}({}_{s}\bar{\boldsymbol{\sigma}}_{(0)}) = {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}^{J}) = {}_{d}({}_{s}\bar{\boldsymbol{\sigma}}) \tag{2.188}$$

Hence, we can write (2.186) as

$${}_{d}({}_{s}\bar{\boldsymbol{\sigma}}^{(0)}) = {}^{0}\bar{\sigma}\big|_{\underline{\Omega}}[I] + 2\eta_{\underline{\Omega}}\big[\bar{D}\big] + \kappa_{\underline{\Omega}}\text{tr}\big[\bar{D}\big] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{2.189}$$

That is, the linear constitutive theory of order one (2.189) for deviatoric Cauchy stress tensor is basis independent.

3. Since the material coefficients $\eta_{\underline{\Omega}}$ and $\kappa_{\underline{\Omega}}$ are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $[\bar{D}]$ in the known configuration $\underline{\Omega}$, they can be defined using power law, Carreau-Yasuda model, Sutherland law, etc. similar to nonpolar thermofluids (see reference [93]).

2.5.6 Simplified constitutive theories for ${}^{(0)}\bar{\boldsymbol{m}}$: compressible

The most general constitutive theory for Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ has been presented in section 2.5.3. Unfortunately this constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ requires forty seven material coefficients. In this section we present simplified constitutive theory that are derived using the general constitutive theory presented in section 2.5.3. We consider these in the following.

2.5.6.1 Constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ without $\{g\}$ as argument tensor: compressible case

In this case

$$[{}^{(0)}\bar{m}] = \left[{}^{(0)}\bar{m}\left(\bar{\rho},\, [{}^{\Theta}D],\, \bar{\theta}\right)\right] \tag{2.190}$$

and $\underset{\sim}{N} = 2$ and $\underset{\sim}{M} = 3$. The generators and invariants are

$$\left[{}^{m}\underset{\sim}{\boldsymbol{G}}^{1}\right] = [{}^{\Theta}D] \quad , \quad \left[{}^{m}\underset{\sim}{\boldsymbol{G}}^{2}\right] = [{}^{\Theta}D]^2 \tag{2.191}$$

$${}^{m}\underset{\sim}{I}^{1} = \text{tr}[{}^{\Theta}D] \quad , \quad {}^{m}\underset{\sim}{I}^{2} = \text{tr}\left([{}^{\Theta}D]^2\right) \quad , \quad {}^{m}\underset{\sim}{I}^{3} = \text{tr}\left([{}^{\Theta}D]^3\right) \tag{2.192}$$

and the constitutive theory ${}^{(0)}\bar{\boldsymbol{m}}$ using the generators invariants (2.191) and (2.192) is given by (2.173) with $\underset{\sim}{N} = 2$ and $\underset{\sim}{M} = 3$. This constitutive theory requires fourteen material coefficients, still too many for practical applications. Explicit form is given by the following after Taylor series expansion of the coefficients in the linear combination about a known configuration $\underline{\Omega}$.

$$
\begin{aligned}
[{}^{(0)}\bar{m}] = {}& {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + {}^{m}\underline{a}_1 \mathrm{tr}[{}^{\Theta}D][I] + {}^{m}\underline{a}_2 \mathrm{tr}[{}^{\Theta}D]^2[I] + {}^{m}\underline{a}_3 \mathrm{tr}[{}^{\Theta}D]^3[I] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \\
& + {}^{m}\underline{b}_1[{}^{\Theta}D] + {}^{m}\underline{b}_2[{}^{\Theta}D]^2 + {}^{m}\underline{c}_{11}\left(\mathrm{tr}[{}^{\Theta}D]\right)[{}^{\Theta}D] + {}^{m}\underline{c}_{12}\left(\mathrm{tr}[{}^{\Theta}D]^2\right)[{}^{\Theta}D] \\
& + {}^{m}\underline{c}_{13}\left(\mathrm{tr}[{}^{\Theta}D]^3\right)[{}^{\Theta}D] + {}^{m}\underline{c}_{21}\left(\mathrm{tr}[{}^{\Theta}D]\right)[{}^{\Theta}D]^2 + {}^{m}\underline{c}_{22}\left(\mathrm{tr}[{}^{\Theta}D]^2\right)[{}^{\Theta}D]^2 \\
& + {}^{m}\underline{c}_{23}\left(\mathrm{tr}[{}^{\Theta}D]^3\right)[{}^{\Theta}D]^2 + {}^{m}\underline{d}_1\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{\Theta}D] + {}^{m}\underline{d}_2\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{\Theta}D]^2
\end{aligned}
\tag{2.193}
$$

The material coefficients in (2.193) are defined by (2.172).

2.5.6.2 Constitutive theory for ${}^{(0)}\boldsymbol{\bar{m}}$ that is quadratic in $[{}^{\Theta}D]$ but independent of $\{g\}$: compressible

We begin with (2.193) and neglect those terms on the right side of (2.193) that are of degree higher than two in the components of $[{}^{\Theta}D]$.

$$
\begin{aligned}
[{}^{(0)}\bar{m}] = {}& {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + {}^{m}\underline{a}_1 \mathrm{tr}[{}^{\Theta}D][I] + {}^{m}\underline{a}_2 \mathrm{tr}[{}^{\Theta}D]^2[I] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \\
& + {}^{m}\underline{b}_1[{}^{\Theta}D] + {}^{m}\underline{b}_2[{}^{\Theta}D]^2 + {}^{m}\underline{c}_{11}\left(\mathrm{tr}[{}^{\Theta}D]\right)[{}^{\Theta}D] \\
& + {}^{m}\underline{d}_1\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{\Theta}D] + {}^{m}\underline{d}_2\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[{}^{\Theta}D]^2
\end{aligned}
\tag{2.194}
$$

This constitutive theory requires eight material coefficients. If we further neglect the product terms in $\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)$ (last two terms on the right side of (2.194)) in (2.194), then we obtain

$$
\begin{aligned}
[{}^{(0)}\bar{m}] = {}& {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + {}^{m}\underline{a}_1 \mathrm{tr}[{}^{\Theta}D][I] + {}^{m}\underline{a}_2 \mathrm{tr}[{}^{\Theta}D]^2[I] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \\
& + {}^{m}\underline{b}_1[{}^{\Theta}D] + {}^{m}\underline{b}_2[{}^{\Theta}D]^2 + {}^{m}\underline{c}_{11}\left(\mathrm{tr}[{}^{\Theta}D]\right)[{}^{\Theta}D]
\end{aligned}
\tag{2.195}
$$

This constitutive theory requires only six material coefficients. The dependence of the material coefficients on the invariants in (2.195) can be modified based on the assumptions used here or can be maintained as originally defined in (2.172).

2.5.6.3 Constitutive theory for ${}^{(0)}\boldsymbol{\bar{m}}$ that is linear in $[{}^{\Theta}D]$ but independent of $\{g\}$: compressible

$$
[{}^{(0)}\bar{m}] = {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + {}^{m}\underline{a}_1 \mathrm{tr}[{}^{\Theta}D][I] + {}^{m}\underline{b}_1[{}^{\Theta}D] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I]
\tag{2.196}
$$

If we denote ${}^{m}\kappa_{\underline{\Omega}} = {}^{m}\underline{a}_1$ and $2({}^{m}\eta)_{\underline{\Omega}} = {}^{m}\underline{b}_1$, then we can write (2.196) as

$$
[{}^{(0)}\bar{m}] = {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + 2({}^{m}\eta)_{\underline{\Omega}}[{}^{\Theta}D] + {}^{m}\kappa_{\underline{\Omega}} \mathrm{tr}[{}^{\Theta}D][I] - {}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I]
\tag{2.197}
$$

the material coefficients ${}^{m}\kappa_{\underline{\Omega}}$ and ${}^{m}\eta_{\underline{\Omega}}$ can be functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $[{}^{\Theta}D]$ in the known configuration $\underline{\Omega}$. The constitutive theory 2.197 is the simplest possible constitutive theory for Cauchy moment tensor ${}^{(0)}\boldsymbol{\bar{m}}$ but permits deformation dependent material coefficients. Thus, here also we can use concepts similar to power law, Carreau-Yasuda model, Sutherland law etc.

that are used for the material coefficients in the constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$.

2.5.7 Simplified constitutive theories for heat vector ${}^{(0)}\bar{\boldsymbol{q}}$: compressible

Much simpler (but with limitations) constitutive theories for ${}^{(0)}\bar{\boldsymbol{q}}$ can be derived if we limit its argument tensors. Consider a constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ using $\bar{\rho}$, $\bar{\theta}$, and $\bar{\boldsymbol{g}}$ as the only argument tensors of ${}^{(0)}\bar{\boldsymbol{q}}$ i.e. consider

$$
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho}, \bar{\theta}, \bar{\boldsymbol{g}}\right) \tag{2.198}
$$

In this case we have only one generator and one invariant (i.e. $\tilde{N} = 1$ and $\tilde{M} = 1$).

$$
\left\{{}^{q}\underset{\sim}{\boldsymbol{G}}^{1}\right\} = \{g\} \quad , \quad {}^{q}\underset{\sim}{I}^{1} = \{g\}^{T}\{g\} \tag{2.199}
$$

Following the general derivation in section 2.5.2 we can write the following for $\tilde{N} = 1$, $\tilde{M} = 1$ using (2.164) and (2.199).

$$
\left\{{}^{(0)}\bar{q}\right\} = -{}^{q}\underline{b}_{1}\{g\} - {}^{q}\underline{c}_{11}\left(\{g\}^{T}\{g\}\right)\{g\} - {}^{q}\underline{d}_{1}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\{g\} \tag{2.200}
$$

The material coefficients in (2.200) are defined by (2.163). This constitutive theory is cubic in $\{g\}$, requires only three material coefficients and is the most general constitutive theory based on (2.198). If we denote ${}^{q}\underline{b}_{1} = k_1|_{\underline{\Omega}}$ and ${}^{q}\underline{c}_{11} = k_2|_{\underline{\Omega}}$ then (2.200) can be written as

$$
\left\{{}^{(0)}\bar{q}\right\} = -\left.k_1\right|_{\underline{\Omega}}\{g\} - \left.k_2\right|_{\underline{\Omega}}\left(\{g\}^{T}\{g\}\right)\{g\} - {}^{q}\underline{d}_{1}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\{g\} \tag{2.201}
$$

If we neglect the $\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)$ term in (2.201) we obtain

$$
\left\{{}^{(0)}\bar{q}\right\} = -\left.k_1\right|_{\underline{\Omega}}\{g\} - \left.k_2\right|_{\underline{\Omega}}\left(\{g\}^{T}\{g\}\right)\{g\} \tag{2.202}
$$

If we assume that $\left\{{}^{(0)}\bar{q}\right\}$ is a linear function of $\{g\}$, then we have

$$
\left\{{}^{(0)}\bar{q}\right\} = -\left.k_1\right|_{\underline{\Omega}}\{g\} \tag{2.203}
$$

Equation (2.203) is Fourier's heat conduction law in which the thermal conductivity $\left.k_1\right|_{\underline{\Omega}}$ can be a function of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}$, and $\left.\left(\{g\}^{T}\{g\}\right)\right|_{\underline{\Omega}}$. We note that the constitutive theories given here are basis independent as $\{g\}$ is basis independent, hence in the theories presented here

$$
\left\{{}^{(0)}\bar{q}\right\} = \left\{\bar{q}^{(0)}\right\} = \left\{\bar{q}_{(0)}\right\} = \left\{{}^{(0)}\bar{q}^{J}\right\} = \{q\} \tag{2.204}
$$

holds. We can also derive the constitutive theory (2.203) using the condition $\{q\}^{T}\{g\} \leq 0$ resulting from the entropy inequality. The derivation is standard and can be found in references [71].

2.6 Constitutive theories for ${}_{d}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$: incompressible

In the case of incompressible internal polar thermofluids

$$\begin{aligned} \bar{\rho} &= \rho_0 = \text{constant} \\ \operatorname{div}(\bar{v}) &= 0 \\ \therefore \quad \operatorname{tr}[\gamma^{(1)}] &= \operatorname{tr}[\gamma_{(1)}] = \operatorname{tr}[{}^{(1)}\gamma^J] = \operatorname{tr}\left[\bar{D}\right] = 0 \\ \det[J] &= 1 \end{aligned} \tag{2.205}$$

Hence, density can be eliminated from the argument tensors of the dependent variables in the constitutive theories. This leads to (2.151) as dependent variables in the constitutive theories and their argument tensors. The constitutive theories for the compressible case, presented in section 2.5, also hold for the incompressible case, but with appropriate modifications based on (2.205). If $[{}^{(1)}\gamma]$ is not part of the constitutive theories for the compressible case (sections 2.5.3, 2.5.6 and 2.5.7), then these theories for the compressible case also hold for the incompressible case provided dependence on $\bar{\rho}$ is removed.

2.7 Closure of mathematical model and comments on constitutive theories

In this mathematical model the dependent variables are (numbers in lower case brackets indicate the count i.e. number of variables):

$$\bar{\rho}(1), \quad \bar{v}_i(3), \quad {}_s\bar{\boldsymbol{\sigma}}^{(0)}(6), \quad {}_a\bar{\boldsymbol{\sigma}}^{(0)}(3), \quad \bar{\boldsymbol{m}}^{(0)}(6), \quad \bar{e}(1), \quad \bar{\theta}(1), \quad \bar{\boldsymbol{q}}(3), \quad \bar{\Phi}(1), \quad \bar{\eta}(1)$$

a total of 26. In these, $\bar{\Phi}$ and $\bar{\eta}$ will be eliminated, $\bar{e}(\bar{\rho}, \bar{\theta})$ i.e. $\bar{e}$ is a function of $\bar{\rho}$ and $\bar{\theta}$ for the most general case of compressible matter, hence $\bar{e}$ is also eliminated. This leaves us with the remaining 23 dependent variables in the mathematical model. We have continuity equation (1), linear momentum equations (3), angular momentum equations (3), energy equation (1) and, from the entropy inequality, we have constitutive theories for ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ (6), $\bar{\boldsymbol{m}}^{(0)}$ (6), and $\bar{\boldsymbol{q}}$, (3), a total of 23 equations, hence this mathematical model will have closure once we have constitutive theories for ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ (6), $\bar{\boldsymbol{m}}^{(0)}$ (6), and $\bar{\boldsymbol{q}}$ (3). Development of the constitutive theory is clearly treatment of matter specific physics. The mathematical model derived here is valid for compressible as well as incompressible fluids.

3. INTERNAL POLAR CONTINUUM THEORY FOR SOLID CONTINUA*

3.1 Notations, definitions, measures and preliminary considerations

We use an overbar to express quantities in the current configuration, i.e. all quantities with overbars are functions of deformed coordinates $\bar{x}_i$ and time t. Quantities without an overbar imply Lagrangian description of the quantities in the current configuration, i.e. these are functions of undeformed coordinates x_i and time t. We use the configuration at time $t = t_0 = 0$, commencement of evolution, to be the reference configuration. Thus, x_i ; $i = 1, 2, 3$ and $\bar{\boldsymbol{x}}$ are the coordinates of a material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. This paper only considers Lagrangian description, hence all measures are expressed in terms of coordinates of the material points in the undeformed configuration (same as reference configuration in the present work) $\boldsymbol{x}$ and time t. We use $[J] = [\frac{\partial\{\bar{x}\}}{\partial\{x\}}]$ to be the Jacobian of deformation. We denote ρ_0 to be the density of the solid matter in the reference configuration, hence it is constant. Φ, θ and η denote the Helmholtz free-energy density, temperature and entropy density.

If the existence of different rotations at the neighboring material points (evident from polar decomposition of the Jacobian of deformation) can result in additional energy storage or dissipation then there must be also coexist moments in the deforming matter. Just like points of application of forces when displaced result in work, the moments moving through rotations result in work as well. Thus, *in the development of the polar continuum theory presented here we consider existence of internal rotations and moments independent of forces and displacements*. Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration (Figure 2.1 (a)) with closed boundary $\partial\underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in cut principle of Cauchy. Consider a tetrahedron T_1 shown in Figure 2.1 (a) such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\underset{\sim}{\bar{V}}$ and $\partial\underset{\sim}{\bar{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are nonorthogonal covariant base vector $\tilde{\boldsymbol{g}}_i$. The plane of the tetrahedron formed by the covariant base vectors are flat but obviously nonorthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on the surface of $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron $\bar{T}_1$ is placed back in the volume $\bar{V}$. Consider deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area also on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short

*Portions of the derivation of the conservation and balance laws presented in this chapter appear in the article "A Polar Continuum Theory for Solid Continua" by K.S. Surana, J.N. Reddy, D. Nunez and M. Powell *Int. J. of Engg. Research & Indu. Appls. (IJERIA)* Vol. 8, No. 2, pp. 77–106 (2015) ©Ascent Journals. Portions of the derivation of the constitutive theories appear in the article "Constitutive Theories for Internal Polar Thermoelastic Solid Continua" by K.S. Surana, M. Powell, and J.N. Reddy *J. of Pure and Applied Mathematics: Advances and Applications* Vol. 14, No. 2 pp. 89–150 (2015) ©Scientific Advances Publishers and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite.

3.1.1 Polar decomposition of the Jacobian of deformation and consideration of local rotations

Polar decomposition of the Jacobian of deformation decomposes deformation into pure stretch and pure rotation. Whether we use left stretch or right stretch, the pure rotation tensor is unique. At each material point with infinitesimal volume surrounding it, the Jacobian of deformation $[J]$ can be decomposed into pure rotation $[R]$ and right stretch tensor $[S_r]$ or left stretch tensor $[S_l]$. $[R]$ is orthogonal and $[S_r]$, $[S_l]$ are symmetric and positive definite. The rotation tensor $[R]$ can equivalently be obtained due to rotations $\boldsymbol{\Theta}$ at the material point. Thus, at every material point, the rotation matrix $[R]$ can be viewed as being due to rotations $\boldsymbol{\Theta}$. If varying rotations at the material points (due to different $[J]$) result in energy storage, then there must be existence of conjugate moments $\boldsymbol{m}$ in the deforming matter, thus the motivation for consideration of $\boldsymbol{\Theta}$ and $\boldsymbol{m}$ in the polar continuum theory presented in this paper.

$$[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = [R][S_r] = [S_l][R] \tag{3.1}$$

$$[R] = [R(\boldsymbol{\Theta})] \tag{3.2}$$

Explicit forms of $\boldsymbol{\Theta}$ that is Θ_{x_1}, Θ_{x_2}, Θ_{x_3} or Θ_1, Θ_2, Θ_3 in terms of antisymmetric part of the displacement gradient tensor establish unique and convenient measures of rotations, hence $[R]$ in (3.2) based on the gradients of deformation field as shown in section 3.1.3 is meritorious.

3.1.2 Rotation gradients and strain gradients

Even though the presence of varying rotations between neighboring material points may influence the energy storage and dissipation in some solid continua, the precise manner in which this occurs is not yet established. All we know at this stage is that just like forces and displacements are work conjugate, the rotations and the moments can also be work conjugate if the deforming matter resists varying rotations between the neighboring material points. Through the derivations of the balance laws presented in section 3.2 we establish that the symmetric part of the rotation gradient tensor is energy conjugate to the moment tensor. Thus, it is fair to say that the polar part of the theory presented here is due to rotation gradients. The purpose of the material presented in this section is to demonstrate that the polar theory presented here is not the same as the strain gradient theories published in the literature.

In reference [95], the author shows a relationship between the gradients of the rotations in terms of gradients of the strain tensor and the rotation tensor. Based on these and other similar works, it is argued and mostly accepted that the continuum theories that incorporate rotation gradients are same as those that are derived using strain gradients in the conservation and balance laws. In section 1.1, 3.1 and 3.1.1 we have explained the physics we propose to incorporate by using rotations in the continuum theory. The purpose of the material that follows is: (i) first to establish a relationship between the gradients of rotations and the gradients of the strain tensor (similar to reference [95]) and (ii) secondly, to demonstrate, using these relations, that the continuum theories based on rotation gradients and those based on strain gradients are in fact not the same. The resulting theories from the two approaches describe different physics. For simplicity, consider a two dimensional state of deformation in x_1x_2-plane. The displacement gradient tensor $[{}^dJ]$ in this case is

$$[{}^dJ] = \frac{\partial\{u_1, u_2\}}{\partial\{x_1, x_2\}} = [{}^d_sJ] + [{}^d_aJ] \tag{3.3}$$

$[{}^d_sJ]$ and $[{}^d_aJ]$ being symmetric and antisymmetric tensors.

$$[{}^d_aJ] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Theta_{x_3} \\ -\Theta_{x_3} & 0 \end{bmatrix} \tag{3.4}$$

in which

$$\Theta_{x_3} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \Theta_3 \tag{3.5}$$

is the rotation about the x_3 axis. Gradients of Θ_{x_3} with respect to x_1 and x_2 are

$$\begin{aligned} \Theta_{3,1} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1^2}\right) \\ \Theta_{3,2} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) \end{aligned} \tag{3.6}$$

For small deformation, the strain measures are

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} \\ \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{3.7}$$

Substituting from (3.7) into (3.6) we can obtain

$$\begin{aligned} \Theta_{3,1} &= \frac{\partial \varepsilon_{11}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_1} \\ \Theta_{3,2} &= \frac{\partial \varepsilon_{12}}{\partial x_2} - \frac{\partial \varepsilon_{22}}{\partial x_1} \end{aligned} \tag{3.8}$$

In (3.8), the gradients $\Theta_{3,1}$ and $\Theta_{3,2}$ of rotation Θ_{x_3} are completely expressed in terms of the gradients of ε_{11} and ε_{22} with respect to x_2 and x_1 and ε_{12} with respect to x_1 as well as x_2.

Remarks

(1) From (3.8) we note that gradients of Θ_{x_3} are functions of $\partial\varepsilon_{11}/\partial x_2$, $\partial\varepsilon_{22}/\partial x_1$, $\partial\varepsilon_{12}/\partial x_1$ and $\partial\varepsilon_{12}/\partial x_2$ but are not dependent on $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$. This is expected due to the fact that $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ are gradients of the elongations per unit length in x_1 and x_2 directions, hence cannot possibly contribute to the gradients of rotations.

(2) Considerations of $\Theta_{3,1}$ and $\Theta_{3,2}$ in the polar theory is identically equivalent to replacing these by the right side of the expressions in (3.8). As long as this condition is satisfied, the polar theory based on rotation gradients is the same as the polar theory based on strain gradients. We keep in mind that $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ are not part of the expressions of rotation gradients in (3.8).

(3) A polar theory based on strain gradients must consider $\varepsilon_{ij,k}$, i.e. gradients of all six strains with respect to x_1, x_2 and x_3. Thus, at the onset, it is clear that the strain gradient polar theory for the 2D case will also consider $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ in the derivation in addition to the other strain gradients that appear in (3.8). This undoubtedly brings in different physics than what is described by (3.8). If we consider three dimensional case (i.e. $\mathbb{R}^3$) then we would find that additionally $\partial\varepsilon_{33}/\partial x_3$ will appear in this strain gradient polar theory but will be absent in the definitions of the gradients of rotations.

(4) The rotation gradient polar theory resulting due to consideration of local rotations is targeted towards specific physics of rotations and rates of rotations resulting in energy storage and dissipation in a deforming solid. *We have shown that the polar theory based on rotation gradients is clearly not the same as the strain gradient theories.* We remark that equation (3.8) representing rotation gradients as a function of some (and not all) of the strain gradients is a consequence of the mathematical manipulation.

3.1.3 Stress, moment and strain tensors and considerations of rotations

Based on the small deformation assumption, the deformed coordinates $\bar{x}_i$ are approximately same as undeformed coordinates x_i, thus the deformed tetrahedron $\bar{T}_1$ in the current configuration is close to its map T_1 in the reference configuration. With this assumption all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for the moment tensors. Thus within the assumption $\bar{\boldsymbol{x}} \simeq \boldsymbol{x}$ we can write

$$\bar{\boldsymbol{P}} = \boldsymbol{P}, \qquad \bar{\boldsymbol{M}} = \boldsymbol{M} \tag{3.9}$$

The Cauchy principle for $\boldsymbol{P}$ and $\boldsymbol{M}$ gives

$$\boldsymbol{P} = \boldsymbol{\sigma}\cdot\boldsymbol{n}, \qquad \boldsymbol{M} = \boldsymbol{m}\cdot\boldsymbol{n} \tag{3.10}$$

in which $\boldsymbol{\sigma}$ is Cauchy stress tensor and $\boldsymbol{m}$ is Cauchy moment tensor (per unit area). The displacement gradient matrix $[{}^dJ]$ and its decomposition into symmetric and antisymmetric parts $[{}^d_sJ]$ and $[{}^d_aJ]$ gives

$${}^dJ_{ij} = \frac{\partial u_i}{\partial x_j} \qquad \text{or} \qquad [{}^dJ] = \frac{\partial\{u\}}{\partial\{x\}} = [{}^d_sJ] + [{}^d_aJ] \tag{3.11}$$

$$\begin{aligned} [{}^d_sJ] &= \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) \\ [{}^d_aJ] &= \frac{1}{2}\left([{}^dJ] - [{}^dJ]^T\right) \end{aligned} \tag{3.12}$$

Let $\{\Theta\} = \begin{bmatrix} \Theta_{x_1} & \Theta_{x_2} & \Theta_{x_3} \end{bmatrix}^T$ or $\boldsymbol{\Theta}$ be the *rotation about* ox_1*,* ox_2 *and* ox_3 *axes of the* x*-frame*, then we have

$$[{}^d_a J] = \begin{bmatrix} 0 & \Theta_{x_3} & -\Theta_{x_2} \\ -\Theta_{x_3} & 0 & \Theta_{x_1} \\ \Theta_{x_2} & -\Theta_{x_1} & 0 \end{bmatrix} \tag{3.13}$$

in which

$$\begin{aligned} \Theta_{x_1} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \Theta_{x_2} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \\ \Theta_{x_3} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{3.14}$$

We define the *gradients of rotation* $\boldsymbol{\Theta}$ by

$$[{}^\Theta J] = \frac{\partial\{\Theta\}}{\partial\{x\}} \qquad \text{or} \qquad {}^\Theta J_{ij} = \frac{\partial \Theta_i}{\partial x_j} \tag{3.15}$$

We also decompose $[{}^\Theta J]$ into symmetric and antisymmetric parts $[{}^\Theta_s J]$ and $[{}^\Theta_a J]$

$$[{}^\Theta J] = [{}^\Theta_s J] + [{}^\Theta_a J] \tag{3.16}$$

in which

$$\begin{aligned} [{}^\Theta_s J] &= \frac{1}{2}\left([{}^\Theta J] + [{}^\Theta J]^T\right) \\ [{}^\Theta_a J] &= \frac{1}{2}\left([{}^\Theta J] - [{}^\Theta J]^T\right) \end{aligned} \tag{3.17}$$

For finite deformation, Green's strain tensor is a suitable choice for measure of strain.

$$[\varepsilon] = \frac{1}{2}\left([J]^T[J] - [I]\right) \tag{3.18}$$

and since

$$[J] = [I] + [{}^d J] \tag{3.19}$$

then $[\varepsilon]$ can be expressed in terms of $[{}^d J]$

$$[\varepsilon] = \frac{1}{2}\left([{}^d J] + [{}^d J]^T + [{}^d J]^T[{}^d J]\right) \tag{3.20}$$

For small deformation, we approximate $[\varepsilon]$ by

$$[\varepsilon] \simeq \frac{1}{2}\left([{}^d J] + [{}^d J]^T\right) = [{}^d_s J] \tag{3.21}$$

and correspondingly, due to rotation, we define $[{}^{\Theta}\varepsilon]$ as

$$[{}^{\Theta}\varepsilon] = \frac{1}{2}\left([{}^{\Theta}J] + [{}^{\Theta}J]^T\right) = [{}^{\Theta}_{s}J] \tag{3.22}$$

We also define the *gradients of velocities* as

$$\frac{\partial\{v\}}{\partial\{x\}} = [L] = [D] + [W] \tag{3.23}$$

in which

$$\begin{aligned} [D] &= \frac{1}{2}\left([L] + [L]^T\right) \\ [W] &= \frac{1}{2}\left([L] - [L]^T\right) \end{aligned} \tag{3.24}$$

Likewise, the *gradients of the rates of rotation* are defined as

$$\frac{\partial\{\dot{\Theta}\}}{\partial\{x\}} = [{}^{\Theta}L] = [{}^{\Theta}D] + [{}^{\Theta}W] \tag{3.25}$$

in which

$$\begin{aligned} [{}^{\Theta}D] &= \frac{1}{2}\left([{}^{\Theta}L] + [{}^{\Theta}L]^T\right) \\ [{}^{\Theta}W] &= \frac{1}{2}\left([{}^{\Theta}L] - [{}^{\Theta}L]^T\right) \end{aligned} \tag{3.26}$$

3.2 Conservation and balance laws

We remark that the polar continuum theory considered here incorporates new physics due to internal varying rotations between the material points. This physics is absent in the currently used thermodynamic framework for isotropic, homogeneous solid continua. This new physics due to rotations may influence some or all conservation and balance laws. In order to determine the precise influence of the new physics (or lack of it) on the conservation and balance laws, we must initiate the derivations of the conservation and balance laws at a fundamental stage as we do for the non-polar case [71] so that the resulting equations can be compared with the non-polar case to determine how these laws are modified or influenced by the physics due to internal varying rotations. We caution that after the derivation of conservation and balance laws we may find that some laws are not influenced by this new physics in which case the corresponding equations will obviously be the same as those for the non-polar case. Nonetheless the derivation of all conservation and balance laws must be presented in completeness otherwise we can not determine whether a particular law is influenced by this new physics when compared to the non-polar case. We wish to remark that in the following sections even if some derivations yield the same equations as for the non-polar case, their derivations are essential to keep in the paper as these are necessary to establish this conclusion compared to the non-polar case.

In a polar continuum theory with displacements, displacement gradients, rotations, and rotation gradients as field variables, we must consider the following conservation and balance laws based on the assumption of thermodynamic equilibrium during the evolution: (1) conservation of mass and conservation of inertia, (2) balance of linear momenta, (3) balance of angular momenta, (4) balance of moments of moments (i.e., couples), (5) first law of thermodynamics (i.e. balance of energy), and (6) second law of thermodynamics (i.e. entropy inequality). We consider details of the derivations of these in the following sections.

3.2.1 Conservation of mass and inertia

The continuity equation resulting from the principle of conservation of mass remains for non-polar continuum remains the same as for the polar case. We obtain the following continuity equation in Lagrangian description [71, 73–75]:

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \tag{3.27}$$

where $\rho_0(\boldsymbol{x})$ is the density in the reference configuration and $\rho(\boldsymbol{x}, t)$ is the Lagrangian description of the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration. In micropolar continuum theories we consider continuum with microfibers. In a deforming volume of matter these microfibers (considered inextensible in micro-polar continuum theories) will have inertial effects due to rotation. Conservation of inertia refers to such inertial effects. *In the polar continuum theory considered here we do not consider the inertial effects.* Thus, that in the polar continuum theory considered here there is only one conservation law leading to same continuity equation (3.27) as in case of non-polar continuum theory.

3.2.2 Balance of linear momenta

For a deforming volume of matter the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is same as that for non-polar continuum theory. Thus, we can write (for small deformation) the following [71]:

$$\begin{gathered}\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0 \\ \text{or} \\ \rho_0 \frac{D\{v\}}{Dt} - \rho_0\{F^b\} - [\sigma]^T\{\nabla\} = 0\end{gathered} \tag{3.28}$$

In Lagrangian description $\frac{D}{Dt} = \frac{\partial}{\partial t}$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ are velocities, $\boldsymbol{F}^b$ are body forces per unit mass and $\boldsymbol{\sigma}$ is the stress tensor. Equations (3.28) are momentum equations in the x_1, x_2, and x_3 directions.

3.2.3 Balance of angular momenta

The principle of balance of angular momentum for a polar continuum can be stated as follows: *The time rate of change of total moment of momenta for a polar continuum is equal to the vector sum of the moments of external forces and the moments.* Thus, due to the surface stress $\bar{\boldsymbol{P}}$, surface moment $\bar{\boldsymbol{M}}$ (per unit area), body force $\bar{\boldsymbol{F}}^b$ (per unit mass) and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using the Eulerian description) we can write the following:

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}}\, d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b\, d\bar{V} \tag{3.29}$$

We consider each terms in (3.29) individually.

$$\begin{aligned}
\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} &= \frac{D}{Dt}\int\limits_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{v}_j\bar{\rho}d\bar{V} \\
&= \frac{D}{Dt}\int\limits_{V} \epsilon_{ijk}x_i v_j \rho_0 dV \\
&= \int\limits_{V} \rho_0\epsilon_{ijk}\frac{D}{Dt}\left(x_i v_j\right) dV \\
&= \int\limits_{V} \rho_0\epsilon_{ijk}\left(v_i v_j + x_i\frac{Dv_j}{Dt}\right) dV
\end{aligned} \tag{3.30}$$

The first term on the right hand side is

$$\begin{aligned}
\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\boldsymbol{n}} - (\bar{\boldsymbol{m}})^T \cdot \bar{\boldsymbol{n}}\right) d\bar{A} \\
&= \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\boldsymbol{n}} d\bar{A} - (\bar{\boldsymbol{m}})^T \cdot \bar{\boldsymbol{n}} d\bar{A}\right) \\
&= \int\limits_{\partial V} \left(\boldsymbol{x} \times (\boldsymbol{\sigma})^T \cdot \boldsymbol{n} dA - (\boldsymbol{m})^T \cdot \boldsymbol{n} dA\right) \\
&= \int\limits_{\partial V} \left(\epsilon_{ijk}x_i\sigma_{mj}n_m - m_{mk}n_m\right) dA
\end{aligned} \tag{3.31}$$

in which $\bar{\boldsymbol{\sigma}}$ is the *Cauchy stress tensor* and $\bar{\boldsymbol{m}}$ is the *Cauchy moment tensor*. Using divergence theorem yields

$$\begin{aligned}
\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_{V} \left(\epsilon_{ijk}\left(x_i\sigma_{mj}\right)_{,m} - m_{mk,m}\right) dV \\
&= \int\limits_{V} \left(\epsilon_{ijk}\left(\delta_{im}\sigma_{mj} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV \\
&= \int\limits_{V} \left(\epsilon_{ijk}\left(\sigma_{ij} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV
\end{aligned} \tag{3.32}$$

The second term on the right hand side is

$$\int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} = \int_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{F}_j^b\bar{\rho}d\bar{V} = \int_V \epsilon_{ijk}x_iF_j{}^b\rho_0 dV \tag{3.33}$$

Substituting from (3.30), (3.31) and (3.33) into (3.29)

$$\begin{aligned}&\int_V \rho_0\epsilon_{ijk}\left(v_iv_j + x_i\frac{Dv_j}{Dt}\right)dV \\ &\quad = \int_V \left(\epsilon_{ijk}(\sigma_{ij} + x_i(\sigma_{mj})_{,m}) - m_{mk,m}\right)dV + \int_V \epsilon_{ijk}x_iF_j^b\rho_0\,dV\end{aligned} \tag{3.34}$$

We note that

$$\epsilon_{ijk}v_iv_j = 0 \tag{3.35}$$

hence, (3.34) reduces to

$$\int_V \epsilon_{ijk}\left(x_i\left(\rho_0\frac{Dv_j}{Dt} - \rho_0F_j^b - \sigma_{mj,m}\right)\right)dV + \int_V (m_{mk,m} - \epsilon_{ijk}\sigma_{ij})\,dV = 0 \tag{3.36}$$

Using balance of linear momenta (3.28) in (3.36) we obtain

$$\int_V (m_{mk,m} - \epsilon_{ijk}\sigma_{ij})\,dV = 0 \tag{3.37}$$

and since the volume V is arbitrary

$$m_{mk,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{3.38}$$

$$\text{or} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.39}$$

$$\text{or} \qquad [m]^T\{\nabla\} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.40}$$

Equation (3.38) represents balance of angular momenta. We note that *the Cauchy stress tensor* $\boldsymbol{\sigma}$ *is non-symmetric*. It is instructive to expand (3.38) into three equations

$$\begin{aligned}\frac{\partial m_{i1}}{\partial x_i} - (\sigma_{23} - \sigma_{32}) = 0 \\ \frac{\partial m_{i2}}{\partial x_i} - (\sigma_{31} - \sigma_{13}) = 0 \\ \frac{\partial m_{i3}}{\partial x_i} - (\sigma_{12} - \sigma_{21}) = 0\end{aligned} \tag{3.41}$$

From (3.41) we note that *off-diagonal elements of the stress tensor* $\boldsymbol{\sigma}$ *are balanced by the gradients of the Cauchy moment tensor.*

Remarks

(a) In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components obviously do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by the shear stresses.

(b) In the case of non-polar solid continua, the balance of angular momenta is a statement of self equilibrating moments due to shear stresses that yields

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.42}$$

which implies that $\boldsymbol{\sigma}$ is symmetric. An important point to note is that (3.42) is a result of stress couples due to shear stresses.

(c) In the case of polar continua, the existence of moments $[m]$ due to the material constitution resisting the rotations results in the shear stress couples being balanced by the internal moments. Thus, for polar continua, the balance of angular momenta yields (3.40) instead of (3.42), i.e.

$$[m]^T\{\nabla\} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.43}$$

We note that (3.43) is also a result of stress couples caused by shear stresses.

(d) Thus, both non-polar and polar continuum theories use stress couples in the angular momenta balance law. *Referring to the polar continuum theory as stress couple theory is inappropriate as the non-polar theory also makes use of the stress couples.*

(e) From (3.40) or (3.41) we note that gradients of $[m]$ equilibrate with the antisymmetric components of the stress tensor $\boldsymbol{\sigma}$ as the symmetric components cancel each other in each of the three equations in (3.41).

(f) Lastly, we emphasize that *appearance of equation* (3.40) *in other theories published in the literature does not necessarily make the polar continuum theory presented here same as those in the literature.* In this work, we begin by demonstrating that the varying rotations at the neighboring material points, when resisted by the deforming matter, require existence of internal moment tensor $[m]$. The balance of angular momenta establishes relationship between $[m]$ and $[\sigma]$ (equations (3.40) or (3.41)).

3.2.4 Balance of moments of moments (or Couples)

Forces, moments, moments of moments . . . are progressively higher order effects or terms, hence must satisfy appropriate balance laws to ensure absence of rigid rotation or rigid translation of the deforming volume of continua. Balance of angular momenta (moments of forces) must be considered for couples created by forces and the moments. Likewise, since moment is similar to force, but is a higher order effect or term than force, a balance law similar to balance of angular momentum i.e. balance of moment of couples or moments must be considered to ensure lack of rigid motion of the deforming continua. Just like in the case of non-polar, isotropic, homogeneous fluent continua balance of angular momenta (moments of forces) restricts the Cauchy stress tensor to be symmetric, we can expect this balance law to impose some restrictions on the Cauchy moment tensor. This argument is similar to that which is presented by Yang et al. [69] in their "modified couple stress theory".

For the deforming volume of matter to be in equilibrium, the moments of moments (or couples) must vanish. In the moment of moments we must consider $\bar{\boldsymbol{M}}$ and also the shear components of the stress tensor $\bar{\boldsymbol{\sigma}}$, i.e., $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}$. Thus, we can write (neglecting inertial terms) in Eulerian description

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}})\, d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0 \tag{3.44}$$

We expand the second term in (3.44) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem.

$$\begin{aligned}
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial\bar{V}} \epsilon_{ijk} x_i \bar{M}_j\, d\bar{A} \\
&= \int_{\partial\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m\, d\bar{A} \\
&= \int_{\bar{V}} (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m}\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{x}_{i,m} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\delta_{im} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int_{\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj,m}\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}\right)\, d\bar{V}
\end{aligned} \tag{3.45}$$

Using equation (3.45) in (3.44) and collecting terms

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}\right) d\bar{V} - \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} = 0 \tag{3.46}$$

The first term in (3.46) vanishes due to balance of angular momenta (3.38) and we obtain

$$\int_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}\, d\bar{V} = 0 \tag{3.47}$$

and since $\bar{V}$ is arbitrary, (3.47) implies

$$\epsilon_{ijk}\bar{m}_{ij} = 0 \qquad \text{and} \qquad \epsilon_{ijk}m_{ij} = 0 \tag{3.48}$$

Equation (3.48) implies that *the Cauchy moment tensor* $\boldsymbol{m}$ *is symmetric*. Thus, we can see that the consequence of this balance law is to impose the restriction of symmetry on the Cauchy moment tensor. We note that in the polar theory presented here, the Cauchy moment tensor is symmetric, but the Cauchy stress tensor is nonsymmetric, whereas in the corresponding non-polar theory, Cauchy stress tensor is symmetric and Cauchy moment tensor is null as the internal rotations are ignored in the theory. Symmetry of the Cauchy moment tensor is a restriction placed on the Cauchy moment tensor due to this balance law.

3.2.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system. Expressing this as a rate equation in Eulerian description we can write

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{3.49}$$

$\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right) d\bar{V} \tag{3.50}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\, d\bar{A} \tag{3.51}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} (\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot\dot{\bar{\boldsymbol{\Theta}}})\, d\bar{A} \tag{3.52}$$

where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force vector per unit mass, $\bar{\boldsymbol{q}}$ is rate of heat. *In* (3.50) *we have neglected rotary inertia.* This is consistent with the assumption used in the derivation of the conservation law in section 3.2.1. Note that the additional term $\bar{\boldsymbol{M}}\cdot\dot{\bar{\boldsymbol{\Theta}}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of rotations. We expand integrals in (3.50)–(3.52). Following reference [71], we can show the following.

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right) d\bar{V} = \int_{V}\left(\rho_0\frac{De}{Dt} + \rho_0\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt} - \rho_0\boldsymbol{F}^b\cdot\boldsymbol{v}\right) dV \tag{3.53}$$

Using

$$
\begin{aligned}
\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} &= \boldsymbol{q} \cdot \boldsymbol{n} dA \\
\bar{\rho} d\bar{V} &= \rho_0 dV \\
d\bar{V} &= |J| dV
\end{aligned} \tag{3.54}
$$

then, applying divergence theorem

$$
-\int\limits_{\partial \bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} = -\int\limits_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n}\, dA = -\int\limits_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV \tag{3.55}
$$

Using stress tensor $\boldsymbol{\sigma}$ and moment tensor $\boldsymbol{m}$ and following reference [71] we can show

$$
\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} d\bar{A} = \boldsymbol{v} \cdot (\boldsymbol{\sigma})^T \cdot \boldsymbol{n} dA = \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} \tag{3.56}
$$

$$
\bar{\boldsymbol{M}} \cdot \dot{\bar{\boldsymbol{\Theta}}} d\bar{A} = \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot \boldsymbol{n} dA = \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A} \tag{3.57}
$$

Thus, we can write the following for (3.49).

$$
\begin{aligned}
&\int\limits_{V} \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = -\int\limits_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int\limits_{\partial V} \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} + \int\limits_{\partial V} \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A}
\end{aligned} \tag{3.58}
$$

and using divergence theorem for the integrals over ∂V

$$
\begin{aligned}
&\int\limits_{V} \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = -\int\limits_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int\limits_{V} \boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) dV + \int\limits_{V} \boldsymbol{\nabla} \cdot \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) dV
\end{aligned} \tag{3.59}
$$

Following reference [71] we can also show that

$$
\boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} \tag{3.60}
$$

$$
\boldsymbol{\nabla} \cdot \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) = \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} \tag{3.61}
$$

and substituting from (3.60) and (3.61) into (3.59)

$$\begin{aligned}&\int\limits_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\ &\qquad = -\int\limits_V \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int\limits_V \left(\boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j}\right) dV\end{aligned} \tag{3.62}$$

Moving all terms to the left of the equality and regrouping

$$\begin{aligned}&\int\limits_V \boldsymbol{v} \cdot \left(\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}\right) dV \\ &\qquad + \int\limits_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) dV = 0\end{aligned} \tag{3.63}$$

Using (3.28) (balance of linear momenta), (3.63) reduces to

$$\int\limits_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) dV = 0 \tag{3.64}$$

Since volume V is arbitrary, we have

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left(m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) = 0 \tag{3.65}$$

We note that in $\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})$, the term $\boldsymbol{\nabla} \cdot \boldsymbol{m}$ can be substituted from (3.39) thereby eliminating gradients of $\boldsymbol{m}$ but introducing $\boldsymbol{\sigma}$ in its place.

3.2.6 Second law of thermodynamics

If $\bar{\eta}$ is entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}(t)$ due to non-contacting bodies, then the rate of increase of entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [71]. Thus

$$\frac{D}{Dt} \int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}\, d\bar{V} \geq \int\limits_{\partial\bar{V}(t)} \bar{h}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{3.66}$$

Using Cauchy's postulate for $\bar{h}$ i.e.,

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{3.67}$$

Using (3.67) in (3.66)

$$\frac{D}{Dt} \int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho} d\bar{V} \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{3.68}$$

We need to transform (3.68) to Lagrangian description. This can be done using

$$
\begin{aligned}
d\bar{V} &= |J| dV \\
\rho_0 &= |J| \bar{\rho} \\
\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} d\bar{A} &= \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA
\end{aligned}
\tag{3.69}
$$

Using (3.69) in (3.68)

$$
\frac{D}{Dt} \int\limits_V \eta \rho_0 \, dV \geq - \int\limits_{\partial V} \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA + \int\limits_V s \rho_0 \, dV \tag{3.70}
$$

Using Gauss's divergence theorem for the terms over ∂V gives (noting that $\boldsymbol{\psi}$ is a tensor of rank one)

$$
\frac{D}{Dt} \int\limits_V \eta \rho_0 \, dV \geq - \int\limits_V \boldsymbol{\nabla} \cdot \boldsymbol{\psi} \, dV + \int\limits_V s \rho_0 \, dV \tag{3.71}
$$

or

$$
\int\limits_v \left(\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s \right) dV \geq 0 \tag{3.72}
$$

and since volume V is arbitrary

$$
\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s \geq 0 \tag{3.73}
$$

Equation (3.73) is entropy inequality and is the most fundamental form resulting from the second law of thermodynamics. A more useful form can be derived if we assume

$$
\boldsymbol{\psi} = \frac{\boldsymbol{q}}{\theta}, \qquad s = \frac{r}{\theta} \tag{3.74}
$$

where θ is absolute temperature, $\boldsymbol{q}$ is the heat vector and r is a suitable potential, then

$$
\boldsymbol{\nabla} \cdot \boldsymbol{\psi} = \psi_{i,i} = \frac{q_{i,i}}{\theta} - \frac{q_i \theta_{,i}}{\theta^2} = \frac{q_{i,i}}{\theta} - \frac{q_i g_i}{\theta^2} = \frac{\boldsymbol{\nabla} \cdot \boldsymbol{q}}{\theta} - \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta^2} \tag{3.75}
$$

Substituting from (3.75) into (3.73) and multiplying throughout by θ yields

$$
\rho_0 \frac{D\eta}{Dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho_0 r) - \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \geq 0 \tag{3.76}
$$

From energy equation (3.65) (after inserting $\rho_0 r$ term)

$$
\boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho_0 r = -\rho_0 \frac{De}{Dt} + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \tag{3.77}
$$

Substituting from (3.77) into (3.76)

$$
\rho_0 \theta \frac{D\eta}{Dt} - \rho_0 \frac{De}{Dt} + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) - \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \geq 0 \tag{3.78}
$$

or

$$\rho_0 \left(\frac{De}{Dt} - \theta \frac{D\eta}{Dt} \right) + \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{3.79}$$

Let Φ be the Helmholtz free energy density defined by

$$\Phi = e - \eta\theta \tag{3.80}$$

$$\therefore \qquad \frac{De}{Dt} - \theta \frac{D\eta}{Dt} = \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \tag{3.81}$$

Substituting from (3.81) into (3.79) we obtain

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{3.82}$$

We note that

$$\sigma_{ji} \frac{\partial v_i}{\partial x_j} = \mathrm{tr} \left([\sigma]^T [\dot{J}]^T \right) = \mathrm{tr} \left([\sigma][\dot{J}] \right) \tag{3.83}$$

$$m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} = \mathrm{tr} \left([m]^T [{}^{\Theta}\dot{J}]^T \right) = \mathrm{tr} \left([m][{}^{\Theta}\dot{J}] \right) \tag{3.84}$$

3.2.7 Complete mathematical model and stress decomposition

The mathematical model derived using conservation of mass, balance of linear and angular momenta, balance of moments of moments (or couples) and first and second laws of thermodynamics is summarized as follows (for small deformation):

$$\rho_0 = |J| \rho \tag{3.85}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 \tag{3.86}$$

$$m_{mk,m} - \epsilon_{ijk} \sigma_{ij} = 0 \tag{3.87}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{3.88}$$

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left(m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) = 0 \tag{3.89}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left(m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) \leq 0 \tag{3.90}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ is non-symmetric (due to (3.87)) whereas the Cauchy moment tensor $\boldsymbol{m}$ is symmetric (due to (3.88)). We decompose Cauchy stress tensor $\boldsymbol{\sigma}$ into symmetric and antisymmetric tensors ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \tag{3.91}$$

and we note that

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} : ({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) = \boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma} \tag{3.92}$$

since

$$\boldsymbol{\epsilon} : {}_s\boldsymbol{\sigma} = 0 \tag{3.93}$$

Using

$$\frac{\partial\{v\}}{\partial\{x\}} = [L] = [D] + [W] \tag{3.94}$$

$$[D] = \frac{1}{2}([L] + [L]^T) \tag{3.95}$$

$$[W] = \frac{1}{2}([L] - [L]^T) \tag{3.96}$$

we obtain

$$\begin{aligned} \sigma_{ji}\frac{\partial v_i}{\partial x_j} &= \sigma_{ji}L_{ij} = ({}_s\sigma_{ji} + {}_a\sigma_{ji})(D_{ij} + W_{ij}) \\ &= {}_s\sigma_{ji}(D_{ij}) + {}_a\sigma_{ji}(W_{ij}) \end{aligned} \tag{3.97}$$

since

$${}_s\sigma_{ji}(W_{ij}) = {}_a\sigma_{ji}(D_{ij}) = 0 \tag{3.98}$$

due to symmetry of ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{D}$. Thus, from (3.97), we can write

$$\operatorname{tr}\left([\sigma]^T[L]^T\right) = \operatorname{tr}([\sigma][L]) = \operatorname{tr}([{}_s\sigma][D]) + \operatorname{tr}([{}_a\sigma][W]) \tag{3.99}$$

Likewise, using

$$\frac{\partial\{\dot{\Theta}\}}{\partial\{x\}} = [{}^\Theta L] = [{}^\Theta D] + [{}^\Theta W] \tag{3.100}$$

$$[{}^\Theta D] = \frac{1}{2}([{}^\Theta L] + [{}^\Theta L]^T) \tag{3.101}$$

$$[{}^\Theta W] = \frac{1}{2}([{}^\Theta L] - [{}^\Theta L]^T) \tag{3.102}$$

we obtain

$$\begin{aligned} m_{ji}\frac{\partial\dot{\Theta}_i}{\partial x_j} &= m_{ji}({}^\Theta L_{ij}) = m_{ji}({}^\Theta D_{ij} + {}^\Theta W_{ij}) \\ &= m_{ji}({}^\Theta D_{ij}) \end{aligned} \tag{3.103}$$

since

$$m_{ji}\left({}^\Theta W_{ij}\right) = 0 \tag{3.104}$$

due to symmetry of $\boldsymbol{m}$. Thus, from (3.103), we can write

$$\mathrm{tr}\left([m]^T[{}^\Theta L]^T\right) = \mathrm{tr}\left([m][{}^\Theta L]\right) = \mathrm{tr}\left([m][{}^\Theta D]\right) \tag{3.105}$$

We also note that by using (3.87), (3.92) and (3.93) we can show that

$$\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) = -\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma}) \tag{3.106}$$

From (3.106), we can substitute in (3.89) and (3.90) if we wish to do so. This substitution eliminates the appearance of the last term in the energy equation (3.89) and the entropy inequality (3.90) but introduce ${}_a\boldsymbol{\sigma}$ instead. Using relations (3.91), (3.92), (3.97) and (3.103), the mathematical model can be written as

$$\rho_0 = |J|\rho \tag{3.107}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot ({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) = 0 \tag{3.108}$$

$$m_{mk,m} - \epsilon_{ijk}({}_a\sigma_{ij}) = 0 \tag{3.109}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{3.110}$$

$$\begin{aligned} \rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - {}_s\sigma_{ji}(D_{ij}) - m_{ji}({}^\Theta D_{ij}) \\ - {}_a\sigma_{ji}(W_{ij}) - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : ({}_a\boldsymbol{\sigma})) = 0 \end{aligned} \tag{3.111}$$

$$\begin{aligned} \rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} - {}_s\sigma_{ji}(D_{ij}) \\ - m_{ji}({}^\Theta D_{ij}) - {}_a\sigma_{ji}(W_{ij}) - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : ({}_a\boldsymbol{\sigma})) \leq 0 \end{aligned} \tag{3.112}$$

A simple calculation by expanding the terms shows that

$$\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma}) = -\mathrm{tr}\left([{}_a\sigma][W]\right) \tag{3.113}$$

By substituting (3.113) in (3.111) and (3.112), the energy equation and entropy inequality reduce to

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \mathrm{tr}\left([{}_s\sigma][D]\right) - \mathrm{tr}\left([m][{}^\Theta D]\right) = 0 \tag{3.114}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} - \mathrm{tr}\left([{}_s\sigma][D]\right) - \mathrm{tr}\left([m][{}^\Theta D]\right) \leq 0 \tag{3.115}$$

Generally we denote

$$\psi_d = \mathrm{tr}\left([{}_s\sigma][D]\right) + \mathrm{tr}\left([m][{}^\Theta D]\right) = {}^s\psi_d + {}^m\psi_d \tag{3.116}$$

where

$$
\begin{aligned}
{}^{s}\psi_d =& \mathrm{tr}\left([{}_{s}\sigma][D]\right) \\
{}^{m}\psi_d =& \mathrm{tr}\left([m][{}^{\Theta}D]\right)
\end{aligned}
\tag{3.117}
$$

in which ψ_d is the dissipation function which is sum of ${}^{s}\psi_d$ and ${}^{m}\psi_d$, the dissipation functions due to ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$. *Equations* (3.107)-(3.110), (3.114) *and* (3.115) *constitute the complete and final mathematical model*. From the energy equation (3.114) and entropy inequality (3.115) *we clearly observe that* $([{}_{s}\sigma], [D])$ *and* $([m], [{}^{\Theta}D])$ *are conjugate pairs*. This conclusion is important in the derivation of the constitutive theories for $[{}_{s}\sigma]$ and $[m]$.

3.3 Alternate forms of the first and second laws of thermodynamics

Since in internal polar thermoelastic solids the rate of external work only results in rate of strain energy, hence does not influence the rate of entropy production, alternate forms of the first and second laws of thermodynamics can be derived in which the rate of strain energy is eliminated. This form of the entropy inequality is truly a statement that contains the rates of entropy as its original intent. Specifically $[{}_{s}\sigma]$ and $[m]$ and their conjugate rates only result in rates of strain energy densities i.e. ${}^{s}\psi_d$ and ${}^{m}\psi_d$ in (3.117) are rates of strain energy associated with $[{}_{s}\sigma]$ and $[m]$.

Let s be the total strain energy, then

$$
\begin{aligned}
{}^{s}\psi_d + {}^{m}\psi_d &= \dot{s} \\
\text{Let} \quad \rho_0 \underset{\sim}{e} &= \rho_0 e - s \\
\rho_0 \underset{\sim}{\Phi} &= \rho_0 \Phi - s
\end{aligned}
\tag{3.118}
$$

In which $\dot{s}$ is the rate of total strain energy, $\underset{\sim}{e}$ and $\underset{\sim}{\Phi}$ are the modified specific internal energy and the modified Helmholtz free energy densities that are free of strain energy density. From (3.118), taking material derivative we obtain

$$
\begin{aligned}
\rho_0 \dot{\underset{\sim}{e}} &= \rho_0 \dot{e} - \dot{s} = \rho_0 \dot{e} - {}^{s}\psi_d - {}^{m}\psi_d \\
\rho_0 \dot{\underset{\sim}{\Phi}} &= \rho_0 \dot{\Phi} - \dot{s} = \rho_0 \dot{\Phi} - {}^{s}\psi_d - {}^{m}\psi_d
\end{aligned}
\tag{3.119}
$$

Substituting $\rho_0 \dot{e} = \rho_0 \dot{\underset{\sim}{e}} + {}^{s}\psi_d + {}^{m}\psi_d$ and $\rho_0 \dot{\Phi} = \rho_0 \dot{\underset{\sim}{\Phi}} + {}^{s}\psi_d + {}^{m}\psi_d$ from (3.119) into (3.114) and (3.115), we obtain

$$
\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \tag{3.120}
$$

$$
\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} \leq 0 \tag{3.121}
$$

Remarks

(1) Using (3.120) and (3.121), we now have (3.107)–(3.110) and (3.120), (3.121) as conservation and balance equations for internal polar thermoelastic solid continua.

(2) Equations (3.120) and (3.121) are completely free of rate of strain energy.

(3) The form of the entropy equality in (3.121) is completely unaffected by the rate of mechanical work as it should be if the rate of mechanical work does not result in rate of entropy production, which is indeed the case in internal polar thermoelastic solids.

(4) We keep in mind that at the onset of the derivation of the entropy inequality and subsequently it is indeed a statement of rates of entropies (see equation (3.47) in reference [96]). It is only after using the energy equation and after introducing the Helmholtz free energy density in the original form of the entropy inequality (equation (3.47) in reference [96]) that we introduce the rate of strain energy in it. The presence of strain energy indeed is out of place in the entropy inequality as substantiated by this observation [71, 96].

(5) The form of the entropy inequality (3.121) establishes that if the entropy inequality is truly a statement that contains only the rates of entropies (as (3.121) does), then it has no mechanism for deriving constitutive theories for $[{}_s\sigma]$ and $[m]$ (established later as dependent variables in the constitutive theories) as for internal polar thermoelastic solid continua these do not influence entropy production.

(6) In (3.115), Φ contains strain energy densities and (3.115) also contains ${}^s\psi_d$ and ${}^s\psi_m$ which are strain energy rates due to $[{}_s\sigma]$ and $[m]$, thus it is not surprising to eventually find that constitutive theories for $[{}_s\sigma]$ and $[m]$ can be derived using Φ as Φ is a function of strain energy due to $[{}_s\sigma]$ and $[m]$. Introducing Φ containing strain energy densities in the entropy inequality is intentional so that constitutive theories for $[{}_s\sigma]$ and $[m]$ are possible using Φ. The fact is that the constitutive theories for $[{}_s\sigma]$ and $[m]$ are related to the corresponding strain energies. The entropy inequality (3.121) clearly shows that if the entropy inequality is purely a statement of the rates of entropy, it contains no mechanism for deriving constitutive theories for $[{}_s\sigma]$ and $[m]$ for internal polar thermoelastic solids.

(7) In this paper we consider both forms of the entropy inequality ((3.115) and (3.121)) in the derivations of the constitutive theories for internal polar thermoelastic solids.

3.4 Derivations of the constitutive theories

In this section we present derivations of the constitutive theories for internal polar thermoelastic solids using conservation and balance equations (3.107)–(3.115) *(approach I)* as well as (3.107)–(3.110) and (3.120) and (3.121) *(approach II)*. The only difference in the two being the choice of energy equation and the entropy inequality, either (3.114) and (3.115) or (3.120) and (3.121).

3.4.1 Approach I

We consider conservation and balance equations (3.107)–(3.115) in the derivation of the constitutive theories presented in this section.

3.4.1.1 Dependent variables in the constitutive theories and their argument tensors

By examining the conservation and balance laws (3.107)–(3.110) it is rather straight forward to conclude the choice of the following as dependent variables in the constitutive theories for internal polar thermoelastic solid continua: Φ, η, $[{}_s\sigma]$, $[m]$ and $\{q\}$. The choices of $\{g\}$ (due to heat vector $\{q\}$) and θ as argument tensors is rather obvious. Since ($[{}_s\sigma]$, $[D]$ or $[{}^{d}_{s}\dot{J}]$) and ($[m]$, $[{}^{\Theta}D]$ or $[{}^{\Theta}_{s}\dot{J}]$) are conjugate due to rate of work, choices of $[{}^{d}_{s}\dot{J}]$ and $[{}^{\Theta}_{s}\dot{J}]$ are necessary as argument tensors. Thus, based on the principle of equipresence [71, 74, 76] we must choose $[{}^{d}_{s}J]$, $[{}^{\Theta}_{s}J]$, $\{g\}$, θ as argument tensors of all dependent variables in the constitutive theories for the internal polar thermoelastic solid continua. Hence, we have

$$
\begin{aligned}
\Phi =& \Phi\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\eta =& \eta\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
{}_s\boldsymbol{\sigma} =& {}_s\boldsymbol{\sigma}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\boldsymbol{m} =& \boldsymbol{m}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\boldsymbol{q} =& \boldsymbol{q}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)
\end{aligned}
\tag{3.122}
$$

However, we know that ($[{}_s\sigma]$, $[{}^{d}_{s}J]$) and ($[m]$, $[{}^{\Theta}_{s}J]$) are work conjugate and that $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ are not work conjugate with $[m]$ and $[{}_s\sigma]$, hence in (3.122), $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ must be eliminated from the argument lists of $[m]$ and $[{}_s\sigma]$. Appearance of both $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ as arguments of Φ and η is essential as Φ contains strain energy density and both $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ are work conjugate to $[{}_s\sigma]$ and $[m]$ i.e. responsible for strain energy.

$$
\begin{aligned}
\Phi =& \Phi\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\eta =& \eta\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
{}_s\boldsymbol{\sigma} =& {}_s\boldsymbol{\sigma}\left([{}^{d}_{s}J], \{g\}, \theta\right)\\
\boldsymbol{m} =& \boldsymbol{m}\left([{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\boldsymbol{q} =& \boldsymbol{q}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)
\end{aligned}
\tag{3.123}
$$

If the stress and strain fields, moment and rotation fields associated with mechanical work are assumed to be independent of $\boldsymbol{g}$, then $\boldsymbol{g}$ can be eliminated from the arguments of ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$. Thus, (3.123) reduce to

$$
\begin{aligned}
\Phi =& \Phi\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
\eta =& \eta\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)\\
{}_s\boldsymbol{\sigma} =& {}_s\boldsymbol{\sigma}\left([{}^{d}_{s}J], \theta\right)\\
\boldsymbol{m} =& \boldsymbol{m}\left([{}^{\Theta}_{s}J], \theta\right)\\
\boldsymbol{q} =& \boldsymbol{q}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)
\end{aligned}
\tag{3.124}
$$

3.4.1.2 Entropy inequality: further considerations

It is more convenient to use $[\varepsilon]$ instead of $[{}_s^d J]$, $[\varepsilon]$ being the linear strain tensor as $[\varepsilon] = [{}_s^d J]$. Hence, (3.124) can be written as

$$
\begin{aligned}
\Phi =& \Phi\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right) \\
\eta =& \eta\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right) \\
{}_s\boldsymbol{\sigma} =& {}_s\boldsymbol{\sigma}\left([\varepsilon], \theta\right) \\
\boldsymbol{m} =& \boldsymbol{m}\left([{}_s^\Theta J], \theta\right) \\
\boldsymbol{q} =& \boldsymbol{q}\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right)
\end{aligned}
\tag{3.125}
$$

Using Φ in (3.125) we can obtain the material derivative of Φ needed in (3.115)

$$
\dot{\Phi} = \frac{\partial \Phi}{\partial \varepsilon_{ki}} \dot{\varepsilon}_{ik} + \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} ({}_s^\Theta \dot{J}_{ik}) + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta}
\tag{3.126}
$$

Substituting $\dot{\Phi}$ from (3.126) in (3.115) and using $[\dot{\varepsilon}] = [{}_s^d \dot{J}]$ we obtain

$$
\rho_0 \left(\frac{\partial \Phi}{\partial \varepsilon_{ki}} \dot{\varepsilon}_{ik} + \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} ({}_s^\Theta \dot{J}_{ik}) + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} - {}_s\sigma_{ki} \dot{\varepsilon}_{ik} - m_{ki} {}_s^\Theta \dot{J}_{ik} \leq 0
\tag{3.127}
$$

Regrouping terms in (3.127)

$$
\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} \right) \dot{\varepsilon}_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} - m_{ki} \right) ({}_s^\Theta \dot{J}_{ik}) + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} + \frac{q_i g_i}{\theta} \leq 0
\tag{3.128}
$$

For the entropy inequality (3.128) to hold for arbitrary but admissible choices of $[\dot{\varepsilon}]$, $[{}_s^\Theta \dot{J}]$, $\dot{\boldsymbol{g}}$ and $\dot{\theta}$, the following must hold.

$$
\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} = 0, \qquad \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} - m_{ki} = 0
\tag{3.129}
$$

$$
\rho_0 \frac{\partial \Phi}{\partial g_i} = 0
\tag{3.130}
$$

$$
\rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) = 0 \quad \text{or} \quad \frac{\partial \Phi}{\partial \theta} + \eta = 0
\tag{3.131}
$$

From (3.130) we conclude that Φ is not a function of $\boldsymbol{g}$ and (3.131) implies that η is deterministic from $\frac{\partial \Phi}{\partial \theta}$, hence η is not a dependent variable in the constitutive theory. From (3.129) we obtain

$$
[{}_s\sigma] = \rho_0 \frac{\partial \Phi}{\partial [\varepsilon]}
\tag{3.132}
$$

$$[m] = \rho_0 \frac{\partial \Phi}{\partial [{}^{\Theta}_{s}J]} \tag{3.133}$$

Thus, if Φ is known as a function of $[\varepsilon]$, then the constitutive theory for $[{}_s\sigma]$ can be derived using (3.132) and if Φ is known as a function of $[{}^{\Theta}_{s}J]$, then the constitutive theory for $[m]$ can be derived using (3.133). In view of (3.129)–(3.131), the entropy inequality (3.128) reduces to

$$\frac{q_i g_i}{\theta} \leq 0 \quad \text{or} \quad q_i g_i \leq 0 \tag{3.134}$$

Inequality (3.134) forms the basis for deriving constitutive theory for the heat vector $\{q\}$. We note there is no mechanism (other than physical reasoning) to remove $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ from the argument list of $\{q\}$ as in (3.125), hence we must maintain the arguments of $\{q\}$ in (3.125).

3.4.1.3 Constitutive theory for ${}_s\boldsymbol{\sigma}$ assuming Φ is a function of the invariants of $\boldsymbol{\varepsilon}$ and θ

Consider

$$[{}_s\sigma] = \rho_0 \frac{\partial \Phi}{\partial [\varepsilon]} \tag{3.135}$$

in which $\Phi = \Phi([\varepsilon], \theta)$. Due to the frame invariance requirement, Φ cannot be a function of $[\varepsilon]$, but instead we must consider Φ as a function of the invariants of $[\varepsilon]$. If we choose the principal invariants of $[\varepsilon]$ i.e. I_ε, II_ε and III_ε [71], then

$$\Phi = \Phi\left(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta\right) \tag{3.136}$$

Using (3.136) in (3.135) it is straightforward to derive [71]

$${}_s\boldsymbol{\sigma} = {}^{\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\underset{\sim}{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\underset{\sim}{\alpha}^2 (\boldsymbol{\varepsilon})^{-1} \tag{3.137}$$

In which

$$\begin{aligned}
{}^{\sigma}\underset{\sim}{\alpha}^0 &= \rho_0 \left(\frac{\partial \Phi}{\partial I_\varepsilon} + \frac{\partial \Phi}{\partial II_\varepsilon} I_\varepsilon \right) \\
{}^{\sigma}\underset{\sim}{\alpha}^1 &= \left(-\rho_0 \frac{\partial \Phi}{\partial II_\varepsilon} \right) \\
{}^{\sigma}\underset{\sim}{\alpha}^{-1} &= \left(\rho_0 \frac{\partial \Phi}{\partial III_\varepsilon} \right)
\end{aligned}$$

Using Hamilton-Cayley theorem [71], (3.136) can be written as

$${}_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2 (\boldsymbol{\varepsilon})^2 \tag{3.138}$$

In which ${}^{\sigma}\tilde{\alpha}^i;\ i = 0, 1, 2$ are functions of ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, 2$ and the invariants I_ε, II_ε, III_ε. Form (3.138) is preferred over (3.137) due to obvious reasons, the absence of $(\boldsymbol{\varepsilon})^{-1}$. This constitutive theory is not usable yet due to the fact that ${}^{\sigma}\tilde{\alpha}^i;\ i = 0,1,2$ are functions of unknown deformation in the current configuration due to the fact that I_ε, II_ε, III_ε and θ are in the current configuration. We postpone further details of determining the material coefficients using (3.138) until a later section. However, (3.138) is a fundamental form of the constitutive theory for ${}_s\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$.

3.4.1.4 Constitutive theory for ${}_s\boldsymbol{\sigma}$ using theory of generators and invariants

Consider

$$
{}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}\left([\varepsilon], \theta\right) \tag{3.139}
$$

${}_s\boldsymbol{\sigma}$ is a symmetric tensor of rank two whose argument tensors are $\boldsymbol{\varepsilon}$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on the theory of generators and invariants [76–92], ${}_s\boldsymbol{\sigma}$ can be expressed as a linear combination of $\boldsymbol{I}$, and the combined generators of its arguments, which in this case are generators of $\boldsymbol{\varepsilon}$ that are symmetric tensors of rank two. Between the argument tensors $\boldsymbol{\varepsilon}$ and θ, the combined generators that are symmetric tensors of rank two are $\boldsymbol{\varepsilon}$ and $(\boldsymbol{\varepsilon})^2$. Using the same coefficients in the linear combination as appears in (3.138), we can write:

$$
{}_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0\boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1\boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2(\boldsymbol{\varepsilon})^2 \tag{3.140}
$$

in which the coefficients ${}^{\sigma}\tilde{\alpha}^i;\ i = 0,1,2$ are functions of I_ε, II_ε, III_ε and θ in the current configuration, i.e.

$$
{}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\left(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta\right);\ i = 0,1,2 \tag{3.141}
$$

We note that (3.140) is the same as (3.138) derived in section 3.4.1.3 with the same definition of the coefficients. Thus, the remarks made in section 3.4.1.3 regarding the coefficients hold here as well. When using the theory of generators and invariants, we can also use the invariants i_ε, ii_ε iii_ε instead of the principal invariants I_ε, II_ε, and III_ε in (3.141). Since the two sets of invariants are related [71], the final outcome remains the same as in section 3.4.1.3.

3.4.1.5 Definition of material coefficients using ${}^{\sigma}\tilde{\alpha}^i;\ i = 0,1,2$ in (3.138) or (3.140)

Consider

$$
{}_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0\boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1\boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2(\boldsymbol{\varepsilon})^2 \tag{3.142}
$$

We consider ${}^{\sigma}\tilde{\alpha}^i;\ i = 0,1,2$ to be functions of I_ε, II_ε, III_ε and temperature θ.

$$
{}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\left(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta\right);\ i = 0,1,2 \tag{3.143}
$$

We can expand ${}^{\sigma}\tilde{\alpha}^i$ in Taylor series in I_ε, II_ε, III_ε, and θ about a known configuration $\underline{\Omega}$. We retain only up to linear terms in the invariants of $\boldsymbol{\varepsilon}$ and temperature θ in the Taylor series expansion.

We introduce the following notation to make the presentation compact:

$$ {}^{\sigma}\underset{\sim}{I}^{1} = I_{\varepsilon}; \quad {}^{\sigma}\underset{\sim}{I}^{2} = II_{\varepsilon}; \quad {}^{\sigma}\underset{\sim}{I}^{3} = III_{\varepsilon} \tag{3.144} $$

Using the notation in (3.144), we can write

$$ {}^{\sigma}\tilde{\alpha}^{i} = {}^{\sigma}\tilde{\alpha}^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{i}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{i}}{\partial \theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \tag{3.145} $$

Substituting from (3.145) into (3.142):

$$ \begin{aligned} {}_{s}\boldsymbol{\sigma} = & \left({}^{\sigma}\tilde{\alpha}^{0}\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{0}}{\partial \theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \right) \boldsymbol{I} \\ & + \left({}^{\sigma}\tilde{\alpha}^{1}\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{1}}{\partial \theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \right) \boldsymbol{\varepsilon} \\ & + \left({}^{\sigma}\tilde{\alpha}^{2}\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{2}}{\partial \theta}\bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \right) (\boldsymbol{\varepsilon})^{2} \end{aligned} \tag{3.146} $$

Collecting coefficients (defined in configuration $\underline{\Omega}$) of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}$, ${}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{\varepsilon}$; $j = 1, 2, 3$, ${}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{\varepsilon}^{2}$; $j = 1, 2, 3$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}$, and $(\theta - \theta_{\underline{\Omega}})(\boldsymbol{\varepsilon})^{2}$, we can write the following using (3.146)

$$ \begin{aligned} {}_{s}\boldsymbol{\sigma} = & \left({}^{\sigma}\tilde{\alpha}^{0}_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) \boldsymbol{I} + \left({}^{\sigma}\tilde{\alpha}^{1}_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) \boldsymbol{\varepsilon} \\ & + \left({}^{\sigma}\tilde{\alpha}^{2}_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \right) (\boldsymbol{\varepsilon})^{2} + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{I}\right) \\ & + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{\varepsilon}\right) + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j}(\boldsymbol{\varepsilon})^{2}\right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{0}}{\partial \theta}\bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}})\boldsymbol{I}\right) \\ & + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{1}}{\partial \theta}\bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}\right) + \frac{\partial\, {}^{\sigma}\tilde{\alpha}^{2}}{\partial \theta}\bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}})(\boldsymbol{\varepsilon})^{2}\right) \end{aligned} \tag{3.147} $$

Let us define

$$
\begin{aligned}
{}^0\bar{\sigma}\big|_{\underline{\Omega}} &= {}^\sigma\underline{b}_0 & {}^\sigma\underline{a}_j &= \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^0}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 \\
{}^\sigma\underline{b}_i &= {}^\sigma\tilde{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{3} \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^i}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; i=0,1,2 & {}^\sigma\underline{c}_{1j} &= \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^1}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 \\
{}^\sigma\underline{c}_{2j} &= \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^2}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 & {}^\sigma\underline{d}_1 &= \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^1}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^\sigma\underline{d}_2 &= \left.\frac{\partial\, {}^\sigma\tilde{\alpha}^2}{\partial\theta}\right|_{\underline{\Omega}} & \underline{\alpha}_{\mathrm{tm}} &= -\left.\frac{\partial\, {}^\sigma\tilde{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned}
\tag{3.148}
$$

Substituting (3.148) into (3.147)

$$
\begin{aligned}
{}_s\boldsymbol{\sigma} = {}& {}^0\bar{\sigma}\big|_{\underline{\Omega}}\boldsymbol{I} + {}^\sigma\underline{b}_1\boldsymbol{\varepsilon} + {}^\sigma\underline{b}_2\boldsymbol{\varepsilon}^2 + \sum_{j=1}^{3} {}^\sigma\underline{a}_j \left({}^\sigma\underset{\sim}{I}^j\boldsymbol{I}\right) + \sum_{j=1}^{3} {}^\sigma\underline{c}_{1j}\left({}^\sigma\underset{\sim}{I}^j\boldsymbol{\varepsilon}\right) \\
&+ \sum_{j=1}^{3} {}^\sigma\underline{c}_{2j}\left({}^\sigma\underset{\sim}{I}^j\boldsymbol{\varepsilon}^2\right) + {}^\sigma\underline{d}_1\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}\right) + {}^\sigma\underline{d}_2\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}^2\right) \\
&+ \underline{\alpha}_{\mathrm{tm}}\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{I}\right)
\end{aligned}
\tag{3.149}
$$

${}^0\bar{\sigma}\big|_{\underline{\Omega}}$ is the initial stress in the configuration $\underline{\Omega}$. This constitutive theory requires determination of 14 material coefficients as defined in (3.148) (excluding ${}^0\bar{\sigma}\big|_{\underline{\Omega}}$), all evaluated in a known configuration $\underline{\Omega}$. The constitutive theory (3.149) for ${}_s\boldsymbol{\sigma}$ is the most general form of the constitutive theory for ${}_s\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$ and temperature θ resulting from the entropy inequality or the theory of generators and invariants. This theory is based on integrity, hence complete, but it contains too many material coefficients to be determined, experimentally or otherwise.

Simplified theory

Here we consider simplifications of the constitutive theory for ${}_s\boldsymbol{\sigma}$ given by (3.149). If we only consider a constitutive theory for ${}_s\boldsymbol{\sigma}$ that is linear in the components of $\boldsymbol{\varepsilon}$ and if we further neglect the $(\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}$ terms, then (3.149) reduces to

$$
{}_s\boldsymbol{\sigma} = {}^0\bar{\sigma}\big|_{\underline{\Omega}}\boldsymbol{I} + {}^\sigma\underline{b}_1\boldsymbol{\varepsilon} + {}^\sigma\underline{a}_1\operatorname{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} + (\underline{\alpha}_{\mathrm{tm}})_{\underline{\Omega}}\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{I}\right)
\tag{3.150}
$$

This constitutive theory only requires three material coefficients ${}^\sigma\underline{a}_1$, ${}^\sigma\underline{b}_1$, and $\underline{\alpha}_{\mathrm{tm}}$, in a known configuration $\underline{\Omega}$.

3.4.1.6 Constitutive theory for moment tensor m assuming Φ is a function of the invariants of $[{}^{\Theta}_{s}J]$ and θ

Consider

$$
[m] = \rho_0 \frac{\partial\Phi}{\partial[{}^{\Theta}_{s}J]}
\tag{3.151}
$$

in which $\Phi = \Phi([{}_{s}^{\Theta}J], \theta)$. Due to frame invariance requirements, Φ can not be a function of $[{}_{s}^{\Theta}J]$ but instead we must consider Φ as a function of the invariants of $[{}_{s}^{\Theta}J]$. If we choose the principle invariants of $[{}_{s}^{\Theta}J]$ i.e. I_Θ, II_Θ, III_Θ [71], then

$$\Phi = \Phi\left(I_\Theta, II_\Theta, III_\Theta, \theta\right) \tag{3.152}$$

using (3.152) in (3.151) it is straightforward to derive

$$[m] = {}^{m}\underset{\sim}{\alpha}^{0}[I] + {}^{m}\underset{\sim}{\alpha}^{1}[{}_{s}^{\Theta}J] + {}^{m}\underset{\sim}{\alpha}^{2}[{}_{s}^{\Theta}J]^{-1} \tag{3.153}$$

in which

$$\begin{gathered} {}^{m}\underset{\sim}{\alpha}^{0} = \rho_0\left(\frac{\partial\Phi}{\partial I_\Theta} + \frac{\partial\Phi}{\partial II_\Theta}I_\Theta\right) \\ {}^{m}\underset{\sim}{\alpha}^{1} = -\rho_0\frac{\partial\Phi}{\partial II_\Theta} \quad ; \quad {}^{m}\underset{\sim}{\alpha}^{-1} = \rho_0\frac{\partial\Phi}{\partial III_\Theta} \end{gathered} \tag{3.154}$$

Using Hamilton-Cayley theorem [71], (3.154) can be written as

$$[m] = {}^{m}\tilde{\alpha}^{0}[I] + {}^{m}\tilde{\alpha}^{1}[{}_{s}^{\Theta}J] + {}^{m}\tilde{\alpha}^{2}[{}_{s}^{\Theta}J]^{2} \tag{3.155}$$

in which ${}^{m}\tilde{\alpha}^{i}$; $i = 0, 1, 2$ are functions of ${}^{m}\underset{\sim}{\alpha}^{i}$; $i = 0, 1, 2$ and the invariants I_Θ, II_Θ, III_θ of $[{}_{s}^{\Theta}J]$. Form (3.155) is preferred over (3.153) due to obvious reasons, the absence of $[{}_{s}^{\Theta}J]^{-1}$. We note that ${}^{m}\tilde{\alpha}^{i}$; $i = 0, 1, 2$ are in the current configuration, hence (3.155) is not usable until the material coefficients are determined using ${}^{m}\tilde{\alpha}^{i}$; $i = 0, 1, 2$. None the less, (3.155) is a fundamental form for the constitutive theory for $[m]$.

3.4.1.7 Constitutive theory for moment tensor m using the theory of generators and invariants

Consider

$$[m] = \left[m\left([{}_{s}^{\Theta}J], \theta\right)\right] \tag{3.156}$$

$[m]$ is a symmetric tensor of rank two whose argument tensors are $[{}_{s}^{\Theta}J]$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on the theory of generators and invariants [76–92], $[m]$ can be expressed as a linear combination of $[I]$ and the combined generators of its argument tensors, which in this case are generators of $[{}_{s}^{\Theta}J]$ that are symmetric tensors of rank two. Between the argument tensors $[{}_{s}^{\Theta}J]$ and θ, the combined generators that are symmetric tensors of rank two are $[{}_{s}^{\Theta}J]$ and $[{}_{s}^{\Theta}J]^2$. Using the same coefficients in the linear combination as those used in (3.155), we can write

$$[m] = {}^{m}\tilde{\alpha}^{0}[I] + {}^{m}\tilde{\alpha}^{1}[{}_{s}^{\Theta}J] + {}^{m}\tilde{\alpha}^{2}[{}_{s}^{\Theta}J]^{2} \tag{3.157}$$

In which the coefficients of ${}^{m}\tilde{\alpha}^{i}$; $i = 0, 1, 2$ are functions of I_Θ, II_Θ, III_θ, and θ in the current configuration. We note that (3.157) is the same as (3.155) derived in section 3.4.1.6 with the definition of coefficients.

3.4.1.8 Determination of material coefficients using ${}^m\tilde{\alpha}^i$; $i = 0, 1, 2$ in (3.155) or (3.157)

Consider

$$[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}^{\Theta}_{s}J] + {}^m\tilde{\alpha}^2[{}^{\Theta}_{s}J]^2 \tag{3.158}$$

in which

$$ {}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\left(I_\Theta, II_\Theta, III_\theta, \theta\right);\; i = 0, 1, 2 \tag{3.159}$$

we expand ${}^m\tilde{\alpha}^i$; $i = 0, 1, 2$ in Taylor series in I_Θ, II_Θ, III_θ, and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants and the temperature θ. We introduce the notation

$$ {}^m\underset{\sim}{I}^1 = I_\Theta, \quad {}^m\underset{\sim}{I}^2 = II_\Theta, \quad \text{and} \quad {}^m\underset{\sim}{I}^3 = III_\Theta \tag{3.160}$$

Using the notation (3.160), the Taylor series expansion yields

$$ {}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \left.\frac{\partial\, {}^m\tilde{\alpha}^i}{\partial\, {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - ({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^m\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right);\; i = 0, 1, 2 \tag{3.161}$$

Substituting (3.161) into (3.158) and collecting coefficients (those defined in $\underline{\Omega}$) of $[I]$, ${}^m\underset{\sim}{I}^j[I]$; $j = 1, 2, 3$, ${}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]$; $j = 1, 2, 3$, ${}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]^2$; $j = 1, 2, 3$, $(\theta - \theta_{\underline{\Omega}})[I]$, $(\theta - \theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]$ and $(\theta - \theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]^2$ and defining

$$\begin{aligned}
{}^0\bar{m}\big|_{\underline{\Omega}} &= {}^m\underline{b}_0 & {}^m\underline{a}_j &= \left.\frac{\partial\, {}^m\tilde{\alpha}^0}{\partial\, {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\; j = 1, 2, 3 \\
{}^m\underline{b}_i = {}^m\tilde{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{3} &\left.\frac{\partial\, {}^m\tilde{\alpha}^i}{\partial\, {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\; i = 0, 1, 2 & {}^m\underline{c}_{1j} &= \left.\frac{\partial\, {}^m\tilde{\alpha}^1}{\partial\, {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\; j = 1, 2, 3 \\
{}^m\underline{c}_{2j} &= \left.\frac{\partial\, {}^m\tilde{\alpha}^2}{\partial\, {}^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\; j = 1, 2, 3 & {}^m\underline{d}_1 &= \left.\frac{\partial\, {}^m\tilde{\alpha}^1}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^m\underline{d}_2 &= \left.\frac{\partial\, {}^m\tilde{\alpha}^2}{\partial\theta}\right|_{\underline{\Omega}} & \underline{\alpha}_{\mathrm{tm}} &= -\left.\frac{\partial\, {}^m\tilde{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{3.162}$$

we can write the following for $[m]$ in (3.158)

$$\begin{aligned}
[m] = {}^0\bar{m}\big|_{\underline{\Omega}}[I] &+ {}^m\underline{b}_1[{}^{\Theta}_{s}J] + {}^m\underline{b}_2[{}^{\Theta}_{s}J]^2 + \sum_{j=1}^{3} {}^m\underline{a}_j\left({}^m\underset{\sim}{I}^j[I]\right) + \sum_{j=1}^{3} {}^m\underline{c}_{1j}\left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]\right) \\
&+ \sum_{j=1}^{3} {}^m\underline{c}_{2j}\left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]^2\right) + {}^m\underline{d}_1\left((\theta - \theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]\right) + {}^m\underline{d}_2\left((\theta - \theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]^2\right) \\
&+ \underline{\alpha}_{\mathrm{tm}}\left((\theta - \theta_{\underline{\Omega}})[I]\right)
\end{aligned} \tag{3.163}$$

This constitutive theory requires determination of 14 material coefficients defined in (3.162), all evaluated in the known configuration $\underline{\Omega}$. Constitutive theory (3.163) is the most general and complete constitutive theory for $[m]$ as it is based on integrity.

A much more simplified constitutive theory for $[m]$ is possible if we only consider a constitutive theory for $[m]$ that is linear in $[{}^{\Theta}_{s}J]$ and if we further neglect the $(\theta - \theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]$ term, then (3.163) reduces to

$$[m] = {}^{0}\bar{m}\big|_{\underline{\Omega}}[I] + {}^{m}\underline{b}_1[{}^{\Theta}_{s}J] + {}^{m}\underline{a}_1 \operatorname{tr}([{}^{\Theta}_{s}J])[I] + \underline{\alpha}_{\mathrm{tm}}\big((\theta - \theta_{\underline{\Omega}})[I]\big) \tag{3.164}$$

This constitutive theory only requires three material coefficients

3.4.2 Approach II

We consider conservation and balance laws (3.107)–(3.110) and (3.120), (3.121) in this derivation. The important aspect of this derivation is that the entropy inequality (3.121) does not contain rate of work due to the fact that for internal polar thermoelastic solids the rate of work does not contribute to rate of entropy production. As a consequence, the entropy inequality (3.121) provides no mechanism for deriving constitutive theories for ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$. That is the constitutive theories for ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$ have no thermodynamic restriction as long as they are derived for isotropic and homogeneous internal polar thermoelastic solid continua as the conservation and balance laws are only valid for this case. Thus, we have complete freedom of deriving the constitutive theories for ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$, a direct consequence of the form of the entropy inequality which is purely a statement of rate of entropies.

3.4.3 Dependent variables in the constitutive theories and their argument tensors

As in the case of approach I, here also it is straightforward to conclude that $\underset{\sim}{\Phi}$, η, $[{}_{s}\sigma]$, $[m]$ and $\{g\}$ is a possible choice of dependent variables in the constitutive theories. Choice of $\{g\}$ and θ as argument tensors is rather obvious. $[{}^{d}_{s}J]$ or $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ must be considered as argument tensors as well due to the fact that these are conjugate with $[{}_{s}\sigma]$ and $[m]$. Thus we have $[\varepsilon]$, $[{}^{\Theta}_{s}J]$, $\{g\}$, and θ as argument tensors. The entropy inequality (3.121) does not contain rate of work, that is the rate of strain energy as for internal polar thermoelastic solid continua the rate of mechanical work can not result in rate of entropy production. Thus $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ can not be argument tensors of $\underset{\sim}{\Phi}$ and η. Furthermore, since $[{}_{s}\sigma]$, $[\varepsilon]$ and $[m]$, $[{}^{\Theta}_{s}J]$ are conjugate pairs, $[{}^{\Theta}_{s}J]$ can not be an argument tensor of $[{}_{s}\sigma]$ and $[\varepsilon]$ can not be an argument tensor of $[m]$. If the stress and strain fields, moment and rotation fields are assumed to be independent of $\{g\}$, then we can arrive at the following for the argument tensors of the dependent variables $\underset{\sim}{\Phi}$, η, $[{}_{s}\sigma]$, $[m]$ and $\{g\}$ in the constitutive theories. $[\varepsilon]$, $[{}^{\Theta}_{s}J]$, $\{g\}$, and θ must be maintained as argument tensors of $\{q\}$ as at this stage there is no mechanism to do otherwise.

$$
\begin{aligned}
\underset{\sim}{\Phi} &= \underset{\sim}{\Phi}\left(\{g\}, \theta\right) \\
\eta &= \eta\left(\{g\}, \theta\right) \\
[{}_s\sigma] &= [{}_s\sigma\left([\varepsilon], \theta\right)] \\
[m] &= \left[m\left([{}_s^{\Theta}J], \theta\right)\right] \\
\{q\} &= \left\{q\left([\varepsilon], [{}_s^{\Theta}J], \{g\}, \theta\right)\right\}
\end{aligned}
\tag{3.165}
$$

3.4.3.1 Entropy inequality, further considerations

We already know that for internal polar thermoelastic solid continua the entropy inequality (3.121) does not contain either of the two conjugate pairs responsible for rate of entropy production (rate of strain energy in this case). Thus in this case the entropy inequality provides no mechanism for deriving constitutive theories of these conjugate quantities for internal polar thermoelastic solid continua. From (3.165) with $\underset{\sim}{\Phi} = \underset{\sim}{\Phi}(\boldsymbol{g}, \theta)$ we can obtain material derivatives of $\underset{\sim}{\Phi}$ needed in (3.121).

$$
\dot{\underset{\sim}{\Phi}} = \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{\partial \underset{\sim}{\Phi}}{\partial \theta} \dot{\theta}
\tag{3.166}
$$

Substituting (3.166) in the entropy inequality (3.121)

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{\partial \underset{\sim}{\Phi}}{\partial \theta} \dot{\theta} + \eta \dot{\theta} \right) + \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{3.167}
$$

or

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta \right) \dot{\theta} + \rho_0 \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{3.168}
$$

For (3.168) to hold for arbitrary but admissible $\dot{\theta}$ and $\dot{\boldsymbol{g}}$ the following must hold.

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta \right) = 0 \quad ; \quad \rho_0 \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} = 0 \quad ; \quad \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{3.169}
$$

Since ρ_0 is constant and $\theta \geq 0$ we can write

$$
\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta = 0
\tag{3.170}
$$

$$
\frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} = 0
\tag{3.171}
$$

$$
\boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{3.172}
$$

Equation (3.170) implies that $\eta = -\frac{\partial \underset{\sim}{\Phi}}{\partial \theta}$ i.e. if $\underset{\sim}{\Phi}$ is known as a function of θ then η is deterministic, thus η can not be a dependent variable in the constitutive theories. Equation (3.171) implies that $\underset{\sim}{\Phi}$

can not be a function of $\boldsymbol{g}$. We note that (3.170)–(3.172) as expected do not provide any mechanism for deriving constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$, however (3.172) can be used to derive constitutive theory for $\boldsymbol{q}$ (shown later). The importance of this derivation is that based on this derivation η is ruled out as a dependent variable in the constitutive theories and it is established that $\underset{\sim}{\Phi}$ is not a function of $\boldsymbol{g}$. Thus, finally (3.165) reduce to the following

$$
\begin{aligned}
\underset{\sim}{\Phi} &= \underset{\sim}{\Phi}\,(\theta) \\
{}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}\left([{}^{d}_{s}J], \theta\right) \\
\boldsymbol{m} &= \boldsymbol{m}\left([{}^{\Theta}_{s}J], \theta\right) \\
\boldsymbol{q} &= \boldsymbol{q}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right)
\end{aligned}
\tag{3.173}
$$

We note that in (3.173) the argument tensors of $\boldsymbol{q}$ remain the same as in (3.165) as there is no additional information or restrictions to do otherwise.

3.5 Constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$: general considerations (approach II)

In this section we consider derivations of the constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$. Constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are derived using the following:

1. Theory of generators and invariants [76–92]
2. Strain energy density function [71]
3. Complementary strain energy density function [71]
4. Using Taylor series expansions [71]

Constitutive theories for $\boldsymbol{q}$ are derived using:

1. Theory of generators and invariants
2. Conditions resulting from the entropy inequality

We consider details of each method in the following sections. First, we make some remarks.

a. We note that both conjugate pairs ($[{}_s\sigma]$, $[{}^{d}_{s}J]$) and ($[m], [{}^{\Theta}_{s}J]$) result in strain energy, hence contribute to the strain energy density function. However, their contributions are independent of each other as established earlier.

b. Remark (a) clearly suggests that the derivations of the constitutive theories for $[{}_s\sigma]$ and $[m]$ are independent of each other.

c. If π is the total strain energy density function, then

$$\pi = {}^{{}_s\sigma}\pi + {}^{m}\pi$$

in which ${}^{s\sigma}\pi$ and ${}^{m}\pi$ are strain energy density functions due to conjugate pairs ($[{}_s\sigma]$, $[{}^d_sJ]$) and ($[m]$, $[{}^{\Theta}_sJ]$). Derivations of the constitutive theories for $[{}_s\sigma]$ and $[m]$ can proceed individually by using ${}^{s\sigma}\pi$ and ${}^{m}\pi$ respectively. The same holds for complementary strain energy density functions (in a later section).

3.5.1 Constitutive theory for ${}_s\boldsymbol{\sigma}$ based on the theory of generators and invariants (Approach II)

In this approach [76–92] we consider ${}_s\boldsymbol{\sigma} = {}_s\,\boldsymbol{\sigma}([{}^d_sJ],\theta) = {}_s\,\boldsymbol{\sigma}([\varepsilon],\theta)$ in which $[\varepsilon] = [{}^d_sJ]$ is the strain tensor for infinitesimal deformation and use the theory of generators and invariants [76–92] to derive constitutive theory for ${}_s\boldsymbol{\sigma}$. In the subsequent derivation it is more convenient to use $[\varepsilon]$ in place of $[{}^d_sJ]$ for the sake of simplicity of notation. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\dots,N$ be the combined generators of the argument tensors $[\varepsilon]$ and θ that are symmetric tensors of rank two and ${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M$ be the combined invariants of the same argument tensors. The generators ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\dots,N$ form an integrity i.e. complete basis. We can now represent $[{}_s\sigma]$ as a linear combination of $[I]$ and ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\dots,N$.

$$[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0[I] + \sum_{i=1}^{N} {}^{\sigma}\tilde{\alpha}^i\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \tag{3.174}$$

in which the coefficients of ${}^{\sigma}\tilde{\alpha}^i; i = 0,1,\dots,N$ are functions of the invariants ${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M$ and θ i.e.

$$ {}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\left({}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M,\theta\right) \tag{3.175}$$

In this particular case we only have two generators and three invariants (i.e. $N = 2$ and $M = 3$)

$$ {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1 = [\varepsilon], \qquad {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^2 = [\varepsilon]^2 \tag{3.176}$$

and

$$ {}^{\sigma}\underset{\sim}{I}^1 = i_\varepsilon \ (\text{or}\ I_\varepsilon); \quad {}^{\sigma}\underset{\sim}{I}^2 = ii_\varepsilon \ (\text{or}\ II_\varepsilon); \quad {}^{\sigma}\underset{\sim}{I}^3 = iii_\epsilon \ (\text{or}\ III_\varepsilon) \tag{3.177}$$

Choice of i_ε, ii_ε, iii_ε or I_ε, II_ε, III_ε (from characteristic equation of $[\varepsilon]$ i.e. principal invariants) does not matter as the two sets of invariants are related. Using (3.176) we can write (3.174) explicitly as follows.

$$[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0[I] + {}^{\sigma}\tilde{\alpha}^1[\varepsilon] + {}^{\sigma}\tilde{\alpha}^2[\varepsilon]^2 \tag{3.178}$$

in which

$$ {}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\left(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta\right) \tag{3.179}$$

Equation (3.178) and (3.179) hold in the current configuration, hence ${}^{\sigma}\tilde{\alpha}^i; i = 1,2$ are unknown as these are functions of the deformation which is not known (yet). Thus, ${}^{\sigma}\tilde{\alpha}^i; i = 0,1,2$ are not material coefficients. We consider Taylor series expansion of each ${}^{\sigma}\tilde{\alpha}^i$ in $I_\varepsilon, II_\varepsilon, III_\varepsilon$ and θ about a known configuration $\underline{\Omega}$ and only retain up to linear terms (for simplicity) in the invariants and the temperature. This is valid based on the principle of smooth neighborhood (assuming ${}^{\sigma}\tilde{\alpha}^i$ are analytic functions of their arguments).

$$\begin{aligned}
{}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\big|_{\underline{\Omega}} + \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial I_\varepsilon}\right|_{\underline{\Omega}} \Big(I_\varepsilon - (I_\varepsilon)_{\underline{\Omega}}\Big) + \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial II_\varepsilon}\right|_{\underline{\Omega}} \Big(II_\varepsilon - (II_\varepsilon)_{\underline{\Omega}}\Big) \\
+ \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial III_\varepsilon}\right|_{\underline{\Omega}} \Big(III_\varepsilon - (III_\varepsilon)_{\underline{\Omega}}\Big) + \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial \theta}\right|_{\underline{\Omega}} \big(\theta - \theta_{\underline{\Omega}}\big)\,; \quad i = 0,1,2
\end{aligned} \tag{3.180}$$

Substituting from (3.180) into (3.179) and collecting coefficients of $[I]$, $[\varepsilon]$, $[\varepsilon]^2$, $I_\varepsilon[I]$, $I_\varepsilon[\varepsilon]$, $I_\varepsilon[\varepsilon]^2$, $II_\varepsilon[I]$, $II_\varepsilon[\varepsilon]$, $II_\varepsilon[\varepsilon]^2$, $III_\varepsilon[I]$, $III_\varepsilon[\varepsilon]$, $III_\varepsilon[\varepsilon]^2$, $(\theta - \theta|_{\underline{\Omega}})[I]$, $(\theta - \theta|_{\underline{\Omega}})[\varepsilon]$, $(\theta - \theta|_{\underline{\Omega}})[\varepsilon]^2$ and defining the following coefficients

$$\begin{aligned}
b_0 &= {}^{\sigma}\tilde{\alpha}^0\big|_{\underline{\Omega}} - b_{01}(I_\varepsilon)_{\underline{\Omega}} - b_{02}(II_\varepsilon)_{\underline{\Omega}} - b_{03}(III_\varepsilon)_{\underline{\Omega}} \\
b_1 &= {}^{\sigma}\tilde{\alpha}^1\big|_{\underline{\Omega}} - b_{11}(I_\varepsilon)_{\underline{\Omega}} - b_{12}(II_\varepsilon)_{\underline{\Omega}} - b_{13}(III_\varepsilon)_{\underline{\Omega}} \\
b_2 &= {}^{\sigma}\tilde{\alpha}^2\big|_{\underline{\Omega}} - b_{21}(I_\varepsilon)_{\underline{\Omega}} - b_{22}(II_\varepsilon)_{\underline{\Omega}} - b_{23}(III_\varepsilon)_{\underline{\Omega}} \\
b_{01} &= \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial I_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{02} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial II_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{03} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial III_\varepsilon}\right|_{\underline{\Omega}} \\
b_{11} &= \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial I_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{12} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial II_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{13} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial III_\varepsilon}\right|_{\underline{\Omega}} \\
b_{21} &= \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial I_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{22} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial II_\varepsilon}\right|_{\underline{\Omega}}, \quad b_{23} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial III_\varepsilon}\right|_{\underline{\Omega}} \\
b_{31} &= \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial \theta}\right|_{\underline{\Omega}}, \quad b_{32} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial \theta}\right|_{\underline{\Omega}}, \quad b_{33} = \left.\frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial \theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{3.181}$$

We can write (3.178) as

$$\begin{aligned}
[{}_s\sigma] = {} & b_0[I] + b_1[\varepsilon] + b_2[\varepsilon]^2 \\
& + b_{01}I_\varepsilon[I] + b_{02}II_\varepsilon[I] + b_{03}III_\varepsilon[I] \\
& + b_{11}I_\varepsilon[\varepsilon] + b_{12}II_\varepsilon[\varepsilon] + b_{13}III_\varepsilon[\varepsilon] \\
& + b_{21}I_\varepsilon[\varepsilon]^2 + b_{22}II_\varepsilon[\varepsilon]^2 + b_{23}III_\varepsilon[\varepsilon]^2 \\
& + b_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + b_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] + b_{33}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon]^2
\end{aligned} \tag{3.182}$$

$b_0, b_1, b_2; b_{ij}$; $i = 0,1,2,3$; $j = 1,2,3$ are material coefficients defined in the known configuration $\underline{\Omega}$. These are functions of the invariants of $[\varepsilon]$ and θ in $\underline{\Omega}$.

The constitutive theory for $[{}_s\sigma]$ defined by (3.182) is based on integrity, hence is complete. It requires fifteen material coefficients and $[{}_s\sigma]$ in (3.182) is up to fifth degree polynomial in the components of $[\varepsilon]$ or displacement gradients, but is linear in temperature θ. Simplified forms of the constitutive theory (3.182) will be considered in a later section.

3.5.2 Constitutive theory for ${}_s\boldsymbol{\sigma}$ using strain energy density function ${}^{s\boldsymbol{\sigma}}\pi$ (Approach II)

Consider the rate of strain energy density function ${}^{s\sigma}\dot{\pi} \equiv \frac{D}{Dt}({}^{s\sigma}\pi)$. If ${}^{s\sigma}\pi$ is the strain energy density function (strain energy per unit mass) due to the conjugate pair $[{}_s\sigma]$ and $[\varepsilon]$, then it's rate ${}^{s\sigma}\dot{\pi} \equiv \frac{D}{Dt}({}^{s\sigma}\pi)$ is given by

$$ {}^{s\sigma}\dot{\pi} = \frac{D}{Dt}\int_V {}^{s\sigma}\pi\rho_0 dV = \frac{D}{Dt}\int_{\bar{V}(t)} {}^{s\boldsymbol{\sigma}}\bar{\pi}\bar{\rho}d\bar{V} \tag{3.183} $$

or

$$ {}^{s\sigma}\dot{\pi} = \int_V \frac{D}{Dt}\left({}^{s\sigma}\pi\right)\rho_0 dV \tag{3.184} $$

We recall that $[{}_s\sigma]$ and $[\varepsilon]$ are energy conjugate and $[{}_s\sigma]$, $[\dot{\varepsilon}]$ are conjugate in rate of energy, hence

$$ {}^{s\sigma}\dot{\pi} = \int_V {}_s\sigma_{ij}\dot{\varepsilon}_{ij} dV \tag{3.185} $$

Using (3.184) and (3.185)

$$ \int_V \left(\rho_0\frac{D}{Dt}\left({}^{s\sigma}\pi\right) - {}_s\sigma_{ij}\dot{\varepsilon}_{ij}\right) dV = 0 \tag{3.186} $$

Since V is arbitrary, we have

$$ \rho_0\frac{D}{Dt}\left({}^{s\sigma}\pi\right) - {}_s\sigma_{ij}\dot{\varepsilon}_{ij} = 0 \tag{3.187} $$

or

$$ \rho_0\frac{\partial}{\partial t}\left({}^{s\sigma}\pi\right) - {}_s\sigma_{ij}\dot{\varepsilon}_{ij} = 0 \tag{3.188} $$

Assuming ${}^{s\sigma}\pi = {}^{s\sigma}\pi([\varepsilon], t)$ we can write (3.188) as

$$ \rho_0\frac{\partial {}^{s\sigma}\pi}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} - {}_s\sigma_{ij}\dot{\varepsilon}_{ij} = 0 \tag{3.189} $$

or

$$ \left(\rho_0\frac{\partial {}^{s\sigma}\pi}{\partial \varepsilon_{ij}} - {}_s\sigma_{ij}\right)\dot{\varepsilon}_{ij} = 0 \tag{3.190} $$

For (3.190) to hold for arbitrary but admissible $[\dot{\varepsilon}]$ the following must hold

$$ {}_s\sigma_{ij} = \rho_0\frac{\partial({}^{s\sigma}\pi)}{\partial \varepsilon_{ij}} \tag{3.191} $$

or

$$ [{}_s\sigma] = [{}_s\sigma]^T = \rho_0\frac{\partial({}^{s\sigma}\pi)}{\partial [\varepsilon]} \tag{3.192} $$

Equation (3.192) can also be derived directly using ${}_s\boldsymbol{\sigma}$ versus $[\varepsilon]$ and constructing the strain energy density function ${}^{s\sigma}\pi$ as

$$
{}^{s\sigma}\pi = \frac{1}{\rho_0}\int_0^{[\varepsilon]} {}_s\sigma_{ij} d\varepsilon_{ij} \tag{3.193}
$$

From (3.193) we can obtain (fundamental theorem of calculus)

$$
[{}_s\sigma] = [{}_s\sigma]^T = \rho_0 \frac{\partial({}^{s\sigma}\pi)}{\partial[\varepsilon]} \tag{3.194}
$$

In the following we derive a constitutive theory for $[{}_s\sigma]$ using (3.194). We consider ${}^{s\sigma}\pi$ as a function of $[\varepsilon]$ and θ, however the principle of frame invariance requires that instead of $[\varepsilon]$ and θ, ${}^{s\sigma}\pi$ must be a function of the invariants of $[\varepsilon]$ and θ. Consider

$$
{}^{s\sigma}\pi = {}^{s\sigma}\pi\left(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta\right) \tag{3.195}
$$

where

$$
\begin{aligned}
I_\varepsilon &= \mathrm{tr}[\varepsilon] = \varepsilon_{ii} \\
II_\varepsilon &= \frac{1}{2}\left(\left(\mathrm{tr}[\varepsilon]\right)^2 - \mathrm{tr}\left([\varepsilon]^2\right)\right) = \frac{1}{2}\varepsilon_{ii}\varepsilon_{kk} - \varepsilon_{ii}\varepsilon_{kk} \\
III_\varepsilon &= \det[\varepsilon]
\end{aligned} \tag{3.196}
$$

Using (3.195) and (3.192) we can write

$$
[{}_s\sigma] = \rho_0\left(\frac{\partial({}^{s\sigma}\pi)}{\partial I_\varepsilon}\frac{\partial I_\varepsilon}{\partial[\varepsilon]} + \frac{\partial({}^{s\sigma}\pi)}{\partial II_\varepsilon}\frac{\partial II_\varepsilon}{\partial[\varepsilon]} + \frac{\partial({}^{s\sigma}\pi)}{\partial III_\varepsilon}\frac{\partial III_\varepsilon}{\partial[\varepsilon]}\right) \tag{3.197}
$$

In the following we determine $\frac{\partial I_\varepsilon}{\partial[\varepsilon]}$, $\frac{\partial II_\varepsilon}{\partial[\varepsilon]}$ and $\frac{\partial III_\varepsilon}{\partial[\varepsilon]}$.
Consider $\frac{\partial I_\varepsilon}{\partial[\varepsilon]}$:

$$
\frac{\partial I_\varepsilon}{\partial \varepsilon_{ij}} = \frac{\partial \varepsilon_{ll}}{\partial \varepsilon_{ij}} = \delta_{ij} \tag{3.198}
$$

or

$$
\frac{\partial I_\varepsilon}{\partial[\varepsilon]} = [I] \tag{3.199}
$$

Consider $\frac{\partial II_\varepsilon}{\partial[\varepsilon]}$: Using (3.196) we can write

$$
\begin{aligned}
\frac{\partial II_\varepsilon}{\partial \varepsilon_{ij}} &= \frac{1}{2}\left(-\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}\varepsilon_{lk} - \varepsilon_{kl}\frac{\partial \varepsilon_{lk}}{\partial \varepsilon_{ij}} + \frac{\partial \varepsilon_{ll}}{\partial \varepsilon_{ij}}\varepsilon_{kk} + \varepsilon_{ll}\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{ij}}\right) \\
&= \frac{1}{2}\left(-\varepsilon_{ij} - \varepsilon_{ij} + \varepsilon_{kk}\delta_{ij} + \varepsilon_{ll}\delta_{ij}\right) \\
&= -\varepsilon_{ij} + \varepsilon_{kk}\delta_{ij}
\end{aligned} \tag{3.200}
$$

Consider $\frac{\partial III_\varepsilon}{\partial[\varepsilon]}$:

$$\frac{\partial III_\varepsilon}{\partial \varepsilon_{ij}} = \frac{\partial(\det[\varepsilon])}{\partial[\varepsilon]} = (\det[\varepsilon])\left[[\varepsilon]^{-1}\right]^T = (\det[\varepsilon])\,[\varepsilon]^{-1} = III_\varepsilon[\varepsilon]^{-1} \tag{3.201}$$

Substituting from (3.199), (3.200) and (3.201) into (3.197)

$$[{}_s\sigma] = \rho_0\left(\frac{\partial({}_s{}^\sigma\pi)}{\partial I_\varepsilon}[I] + \frac{\partial({}_s{}^\sigma\pi)}{\partial II_\varepsilon}\left(-[\varepsilon] - I_\varepsilon[I]\right) + \frac{\partial({}_s{}^\sigma\pi)}{\partial III_\varepsilon}III_\varepsilon[\varepsilon]^{-1}\right) \tag{3.202}$$

or

$$[{}_s\sigma] = \rho_0\left(\frac{\partial({}_s{}^\sigma\pi)}{\partial I_\varepsilon} + \frac{\partial({}_s{}^\sigma\pi)}{\partial II_\varepsilon}I_\varepsilon\right)[I] + \left(-\rho_0\frac{\partial({}_s{}^\sigma\pi)}{\partial II_\varepsilon}\right)[\varepsilon] + \left(\rho_0\frac{\partial({}_s{}^\sigma\pi)}{\partial III_\varepsilon}III_\varepsilon\right)[\varepsilon]^{-1} \tag{3.203}$$

Let

$$\begin{aligned}
{}^\sigma\underset{\sim}{\alpha}^0 &= \rho_0\left(\frac{\partial({}_s{}^\sigma\pi)}{\partial I_\varepsilon} + \frac{\partial({}_s{}^\sigma\pi)}{\partial II_\varepsilon}I_\varepsilon\right)\\
{}^\sigma\underset{\sim}{\alpha}^1 &= -\rho_0\frac{\partial({}_s{}^\sigma\pi)}{\partial II_\varepsilon}\\
{}^\sigma\underset{\sim}{\alpha}^{-1} &= \rho_0\frac{\partial({}_s{}^\sigma\pi)}{\partial III_\varepsilon}III_\varepsilon
\end{aligned} \tag{3.204}$$

Using (3.204) in (3.203), we can write

$$[{}_s\sigma] = {}^\sigma\underset{\sim}{\alpha}^0[I] + {}^\sigma\underset{\sim}{\alpha}^1[\varepsilon] + {}^\sigma\underset{\sim}{\alpha}^{-1}[\varepsilon]^{-1} \tag{3.205}$$

Recall the Hamilton-Cayley theorem [71]

$$[\varepsilon]^3 - I_\varepsilon[\varepsilon]^2 + II_\varepsilon[\varepsilon] - III[I] = 0 \tag{3.206}$$

For non-singular $[\varepsilon]$ i.e. $III_\varepsilon \neq 0$, we can solve (3.206) for $[\varepsilon]^{-1}$ to obtain

$$[\varepsilon]^{-1} = \frac{1}{III_\varepsilon}\left([\varepsilon]^2 - I_\varepsilon[\varepsilon] + II_\varepsilon[I]\right) \tag{3.207}$$

Substituting from (3.207) into (3.205)

$$[{}_s\sigma] = {}^\sigma\underset{\sim}{\alpha}^0[I] + {}^\sigma\underset{\sim}{\alpha}^1[\varepsilon] + \frac{{}^\sigma\underset{\sim}{\alpha}^{-1}}{III_\varepsilon}\left([\varepsilon]^2 - I_\varepsilon[\varepsilon] + II_\varepsilon[I]\right) \tag{3.208}$$

Collecting coefficients of $[I]$, $[\varepsilon]$, and $[\varepsilon]^2$ and defining

$$
\begin{aligned}
{}^{\sigma}\tilde{\alpha}^0 &= {}^{\sigma}\underset{\sim}{\alpha}^0 + \frac{{}^{\sigma}\underset{\sim}{\alpha}^{-1} \mathit{II}_{\varepsilon}}{\mathit{III}_{\varepsilon}} \\
{}^{\sigma}\tilde{\alpha}^1 &= {}^{\sigma}\underset{\sim}{\alpha}^1 - \frac{{}^{\sigma}\underset{\sim}{\alpha}^{-1} I_{\varepsilon}}{\mathit{III}_{\varepsilon}} \\
{}^{\sigma}\tilde{\alpha}^2 &= \frac{{}^{\sigma}\underset{\sim}{\alpha}^{-1}}{\mathit{III}_{\varepsilon}}
\end{aligned}
\tag{3.209}
$$

We can write (3.208) as

$$
[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0 [I] + {}^{\sigma}\tilde{\alpha}^1 [\varepsilon] + {}^{\sigma}\tilde{\alpha}^2 [\varepsilon]^2 \tag{3.210}
$$

Since ${}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i(I_{\varepsilon}, \mathit{II}_{\varepsilon}, \mathit{III}_{\varepsilon}, \theta)$; $i = 0, 1, 2$ we can conclude from (3.210) that ${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i(I_{\varepsilon}, \mathit{II}_{\varepsilon}, \mathit{III}_{\varepsilon}, \theta)$; $i = 0, 1, 2$. The constitutive theory (3.210) is the same as the one derived using the theory of generators and invariants, thus determination of the material coefficients follows the same procedure as used in section 3.5.1 and finally we obtain the same constitutive theory as in section 3.5.1 (equation 3.182) with the same definition of material coefficients.

3.5.3 Constitutive theory for $[\varepsilon]$ in terms of $[{}_s\sigma]$ based on complementary strain energy density function ${}^{s\sigma}\pi^c$ (Approach II)

Similar to the material in section 3.5.2 we begin with the integral defined by (using work conjugate pair $[{}_s\sigma], [\varepsilon]$)

$$
{}^{s\sigma}\pi^c = \frac{1}{\rho_0} \int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij} \, d({}_s\sigma_{ij}) \tag{3.211}
$$

in which ${}^{s\sigma}\pi^c$ is the complementary strain energy density function. From (3.211) we can obtain (fundamental theorem of calculus)

$$
[\varepsilon]^T = [\varepsilon] = \rho_0 \frac{\partial({}^{s\sigma}\pi^c\,({}_s\boldsymbol{\sigma}))}{\partial[{}_s\sigma]} \tag{3.212}
$$

The complementary strain energy density function ${}^{s\sigma}\pi^c$ and the strain energy density function ${}^{s\sigma}\pi$ are obviously related. Consider

$$
\frac{1}{\rho_0} {}_s\sigma_{ij} \varepsilon_{ij} = \frac{1}{\rho_0} \int_0^{\varepsilon} {}_s\sigma_{ij} \, d\varepsilon_{ij} + \frac{1}{\rho_0} \int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij} \, d({}_s\sigma_{ij}) \tag{3.213}
$$

or

$$
\frac{1}{\rho_0} {}_s\sigma_{ij} \varepsilon_{ij} = {}^{s\sigma}\pi([\varepsilon]) + {}^{s\sigma}\pi^c({}_s\boldsymbol{\sigma}) \tag{3.214}
$$

In the case of linear elasticity

$$
{}^{s\sigma}\pi = \frac{1}{\rho_0} \int_0^{\boldsymbol{\varepsilon}} {}_s\sigma_{ij} \, d\varepsilon_{ij} = \frac{1}{2\rho_0} {}_s\sigma_{ij} \varepsilon_{ij} \tag{3.215}
$$

$$ {}^{{}_s\sigma}\pi^c = \frac{1}{\rho_0}\int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij} d_s\sigma_{ij} = \frac{1}{2\rho_0}{}_s\sigma_{ij}\varepsilon_{ij} \tag{3.216} $$

$$ \therefore \quad {}^{{}_s\sigma}\pi = {}^{{}_s\sigma}\ \pi^c \tag{3.217} $$

Using (3.212) we can derive a constitutive theory for $[\varepsilon]$ if we know ${}^{{}_s\sigma}\pi^c$ as a function of ${}_s\boldsymbol{\sigma}$. Consider

$$ [\varepsilon] = [\varepsilon]\left(I_{{}_s\boldsymbol{\sigma}}, II_{{}_s\boldsymbol{\sigma}}, III_{{}_s\boldsymbol{\sigma}}, \theta\right) \tag{3.218} $$

In which

$$ \begin{aligned} I_{{}_s\sigma} &= \mathrm{tr}[{}_s\sigma] = {}_s\sigma_{ii} \\ II_{{}_s\sigma} &= \frac{1}{2}\Big((\mathrm{tr}[{}_s\sigma])^2 - \mathrm{tr}\left([{}_s\sigma]^2\right)\Big) \\ III_{{}_s\sigma} &= \det[{}_s\sigma] \end{aligned} \tag{3.219} $$

Using (3.218) and (3.212) we can write

$$ [{}_s\sigma] = \rho_0\left(\frac{\partial({}^{{}_s\sigma}\pi)}{\partial I_{{}_s\sigma}}\frac{\partial I_{{}_s\sigma}}{\partial[{}_s\sigma]} + \frac{\partial({}^{{}_s\sigma}\pi)}{\partial II_{{}_s\sigma}}\frac{\partial II_{{}_s\sigma}}{\partial[{}_s\sigma]} + \frac{\partial({}^{{}_s\sigma}\pi)}{\partial III_{{}_s\sigma}}\frac{\partial III_{{}_s\sigma}}{\partial[{}_s\sigma]}\right) \tag{3.220} $$

Using (3.219) we can obtain

$$ \frac{\partial I_{{}_s\sigma}}{\partial[{}_s\sigma]} = [I] \tag{3.221} $$

$$ \frac{\partial II_{{}_s\sigma}}{\partial[{}_s\sigma]} = -[{}_s\sigma] + I_{{}_s\sigma}[I] \tag{3.222} $$

$$ \frac{\partial III_{{}_s\sigma}}{\partial[{}_s\sigma]} = III_{{}_s\sigma}[{}_s\sigma]^{-1} \tag{3.223} $$

Substituting (3.221)–(3.223) into (3.220), defining

$$ \begin{aligned} {}^{\varepsilon}\underset{\sim}{\alpha}^0 &= \rho_0\left(\frac{\partial({}^{{}_s\sigma}\pi^c)}{\partial I_{{}_s\sigma}} + \frac{\partial({}^{{}_s\sigma}\pi^c)}{\partial II_{{}_s\sigma}}I_{{}_s\sigma}\right) \\ {}^{\varepsilon}\underset{\sim}{\alpha}^1 &= -\rho_0\frac{\partial({}^{{}_s\sigma}\pi^c)}{\partial II_{{}_s\sigma}} \\ {}^{\varepsilon}\underset{\sim}{\alpha}^{-1} &= \rho_0\frac{\partial({}^{{}_s\sigma}\pi^c)}{\partial III_{{}_s\sigma}}III_{{}_s\sigma} \end{aligned} \tag{3.224} $$

Collecting coefficients of $[I]$, $[{}_s\sigma]$, $[{}_s\sigma]^{-1}$ and using (3.224) we can write

$$ [\varepsilon] = {}^{\varepsilon}\underset{\sim}{\alpha}^0[I] + {}^{\varepsilon}\underset{\sim}{\alpha}^1[{}_s\sigma] + {}^{\varepsilon}\underset{\sim}{\alpha}^{-1}[{}_s\sigma]^{-1} \tag{3.225} $$

$[{}_s\sigma]^{-1}$ in (3.225) can be obtained in terms of $[I]$, $[{}_s\sigma]$, and $[{}_s\sigma]^2$ and the invariants of $[{}_s\sigma]$ using the Hamilton-Cayley theorem to obtain

$$[\varepsilon] = {}^{\varepsilon}\tilde{\alpha}^0[I] + {}^{\varepsilon}\tilde{\alpha}^1[{}_s\sigma] + {}^{\varepsilon}\tilde{\alpha}^2[{}_s\sigma]^2 \tag{3.226}$$

in which

$${}^{\varepsilon}\tilde{\alpha}^i = {}^{\varepsilon}\tilde{\alpha}^i \left(I_{{}_s\sigma}, II_{{}_s\sigma}, III_{{}_s\sigma}, \theta\right); \; i = 0,1,2 \tag{3.227}$$

Since ${}^{\varepsilon}\tilde{\alpha}^i$; $i = 0,1,2$ are functions of ${}^{\varepsilon}\underset{\sim}{\alpha}^i$; $i = -1,0,1$ and ${}^{\varepsilon}\underset{\sim}{\alpha}^i$ are functions of $I_{{}_s\sigma}, II_{{}_s\sigma}, III_{{}_s\sigma}$, and θ, (3.227) holds.

Material coefficients in (3.226) are derived using exactly the same approach as used in section 3.5.1, which would lead to a constitutive theory for $[\varepsilon]$ similar to that for $[{}_s\sigma]$ in equation (3.182). Derivation is straight forward.

3.5.4 Constitutive theory for $[{}_s\sigma]$ using strain energy density function ${}^{{}_s\sigma}\pi([\varepsilon],\theta)$ and expanding it in Taylor series about a known configuration $\underline{\Omega}$ (Approach II)

Consider ${}^{{}_s\sigma}\pi = {}^{{}_s\sigma}\pi([\varepsilon],\theta)$ and expand ${}^{{}_s\sigma}\pi$ in $[\varepsilon]$ about a known configuration $\underline{\Omega}$ using Taylor series [71].

$$\begin{aligned} {}^{{}_s\sigma}\pi = {}& {}^{{}_s\sigma}\pi|_{\underline{\Omega}} + \frac{\partial({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right) + \frac{1}{2!}\left.\frac{\partial^2({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}}\right|_{\underline{\Omega}}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right) \\ & + \frac{1}{3!}\left.\frac{\partial^3({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{pq}}\right|_{\underline{\Omega}}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq} - (\varepsilon_{pq})_{\underline{\Omega}}\right) + \cdots \end{aligned} \tag{3.228}$$

Let

$$\begin{aligned} {}^{{}_s\sigma}\pi|_{\underline{\Omega}} &= C \\ \left.\frac{\partial({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}}\right|_{\underline{\Omega}} &= C_{ij} \\ \left.\frac{\partial^2({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}}\right|_{\underline{\Omega}} &= \hat{C}_{ijkl} \\ \left.\frac{\partial^3({}^{{}_s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{pq}}\right|_{\underline{\Omega}} &= \tilde{C}_{ijklpq} \end{aligned} \tag{3.229}$$

Substituting from (3.229) into (3.228)

$$\begin{aligned} {}^{{}_s\sigma}\pi = {}& C + C_{ij}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right) + \hat{C}_{ijkl}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right) \\ & + \tilde{C}_{ijklpq}\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq} - (\varepsilon_{pq})_{\underline{\Omega}}\right) + \cdots \end{aligned} \tag{3.230}$$

Substituting ${}^{{}_s\sigma}\pi$ from (3.230) into (3.194) and differentiating ${}^{{}_s\sigma}\pi$ with respect to $[\varepsilon]$ and noting that partial derivatives of (3.229) with respect to $[\varepsilon]$ are zero and that

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon_{mn}} \left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right) = \delta_{im}\delta_{jn} \\
&\frac{\partial}{\partial \varepsilon_{mn}} \left(\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\right) \\
&\qquad = \delta_{im}\delta_{jn}\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right) + \left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln} \\
&\frac{\partial}{\partial \varepsilon_{mn}} \left(\left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq} - (\varepsilon_{pq})_{\underline{\Omega}}\right)\right) \\
&\qquad = \delta_{im}\delta_{jn}\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq} - (\varepsilon_{pq})_{\underline{\Omega}}\right) \\
&\qquad + \left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln}\left(\varepsilon_{pq} - (\varepsilon_{pq})_{\underline{\Omega}}\right) \\
&\qquad + \left(\varepsilon_{ij} - (\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl} - (\varepsilon_{kl})_{\underline{\Omega}}\right)\delta_{pm}\delta_{qn}
\end{aligned}
\tag{3.231}
$$

We obtain the following (note that ρ_0 is absorbed in the coefficients in (3.232)).

$$
{}_s\sigma_{mn} = ({}_s\sigma_{mn})_{\underline{\Omega}} + C_{mnij}\varepsilon_{ij} + \bar{\bar{C}}_{mnijkl}\varepsilon_{ij}\varepsilon_{kl} + \cdots \tag{3.232}
$$

In obtaining (3.232) we collect those terms that are defined in the known configuration $\underline{\Omega}$ to define coefficients of $[\varepsilon]$ and $[\varepsilon]^2$ and we use symmetry of the coefficients [71] i.e. $\hat{C}_{mnij} = \hat{C}_{ijmn} \ldots$ etc.

3.5.5 Constitutive theory of $[\varepsilon]$ using complementary strain energy density function ${}^{s\sigma}\pi^c([{}_s\sigma], \theta)$ and expanding it in Taylor series about a known configuration (Approach II)

Considering ${}^{s\sigma}\pi^c = {}^{s\sigma}\ \pi^c([{}_s\sigma], \theta)$ and expanding this in Taylor series in $[{}_s\sigma]$ about a known configuration $\underline{\Omega}$, then using (3.212) and following exactly the same procedure as in section 3.5.4 we can derive a constitutive theory for $[\varepsilon]$ as a function of $[{}_s\sigma]$ that is exactly parallel to (3.232). Details are straight forward and hence omitted for the sake of brevity.

3.5.6 Constitutive theory for $[m]$ based on the theory of generators and invariants (Approach II)

Following section 3.5.1 we consider $[m] = [m([{}^{\Theta}_{s}J], \theta)]$ in which $[{}^{\Theta}_{s}J]$ is the symmetric part of the rotation gradient tensor [96]. $[m]$ and $[{}^{\Theta}_{s}J]$ are symmetric tensors of rank tow and θ is a tensor of rank zero. The combined generators of $[{}^{\Theta}_{s}J]$ and θ that are symmetric tensors of ranks two are $[{}^{\Theta}_{s}J]$ and $[{}^{\Theta}_{s}J]^2$. Thus, we can express $[m]$ as a linear combination of $[I]$, $[{}^{\Theta}_{s}J]$, and $[{}^{\Theta}_{s}J]^2$.

$$
[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}^{\Theta}_{s}J] + {}^m\tilde{\alpha}^2[{}^{\Theta}_{s}J]^2 \tag{3.233}
$$

in which ${}^m\tilde{\alpha}^i$; $i = 0, 1, 2$ are functions of $I_\Theta, II_\Theta, III_\Theta$ and temperature θ, where $I_\Theta, II_\Theta, III_\Theta$ are the principal invariants of $[{}^{\Theta}_{s}J]$ based on the characteristic equation of $[{}^{\Theta}_{s}J]$ i.e.

$$
{}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\left(I_\Theta, II_\Theta, III_\Theta, \theta\right) \tag{3.234}
$$

Equations (3.233) and (3.234) hold in the current configuration. Using (3.233) and (3.234) we define material coefficients. We expand ${}^m\tilde{\alpha}^i; i = 0, 1, 2$ in Taylor series in $I_\Theta, II_\Theta, III_\Theta$ and θ about a known configuration $\underline{\Omega}$ and only retain up to linear terms in the invariants and the temperature.

$$
\begin{aligned}
{}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\big|_{\underline{\Omega}} &+ \frac{\partial\, {}^m\tilde{\alpha}^i}{\partial I_\Theta}\bigg|_{\underline{\Omega}} \Big(I_\Theta - (I_\Theta)_{\underline{\Omega}}\Big) + \frac{\partial\, {}^m\tilde{\alpha}^i}{\partial II_\Theta}\bigg|_{\underline{\Omega}} \Big(II_\Theta - (II_\Theta)_{\underline{\Omega}}\Big) \\
&+ \frac{\partial\, {}^m\tilde{\alpha}^i}{\partial III_\Theta}\bigg|_{\underline{\Omega}} \Big(III_\Theta - (III_\Theta)_{\underline{\Omega}}\Big) + \frac{\partial\, {}^m\tilde{\alpha}^i}{\partial \theta}\bigg|_{\underline{\Omega}} \big(\theta - \theta_{\underline{\Omega}}\big)\,; \quad i = 0,1,2
\end{aligned}
\tag{3.235}
$$

Substituting from (3.235) into (3.234) and collecting coefficients of $[I]$, $[{}^\Theta_s J]$, $[{}^\Theta_s J]^2$, $I_\Theta[I]$, $I_\Theta[{}^\Theta_s J]$, $I_\Theta[{}^\Theta_s J]^2$, $II_\Theta[I]$, $II_\Theta[{}^\Theta_s J]$, $II_\Theta[{}^\Theta_s J]^2$, $III_\Theta[I]$, $III_\Theta[{}^\Theta_s J]$, $III_\Theta[{}^\Theta_s J]^2$, $(\theta - \theta|_{\underline{\Omega}})[I]$, $(\theta - \theta|_{\underline{\Omega}})[{}^\Theta_s J]$, $(\theta - \theta|_{\underline{\Omega}})[{}^\Theta_s J]^2$ and defining the following coefficients

$$
\begin{aligned}
\tilde{b}_0 &= {}^m\tilde{\alpha}^0\big|_{\underline{\Omega}} - \tilde{b}_{01}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{02}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{03}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_1 &= {}^m\tilde{\alpha}^1\big|_{\underline{\Omega}} - \tilde{b}_{11}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{12}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{13}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_2 &= {}^m\tilde{\alpha}^2\big|_{\underline{\Omega}} - \tilde{b}_{21}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{22}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{23}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_{01} &= \frac{\partial\, {}^m\tilde{\alpha}^0}{\partial I_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{02} = \frac{\partial\, {}^m\tilde{\alpha}^0}{\partial II_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{03} = \frac{\partial\, {}^m\tilde{\alpha}^0}{\partial III_\Theta}\bigg|_{\underline{\Omega}} \\
\tilde{b}_{11} &= \frac{\partial\, {}^m\tilde{\alpha}^1}{\partial I_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{12} = \frac{\partial\, {}^m\tilde{\alpha}^1}{\partial II_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{13} = \frac{\partial\, {}^m\tilde{\alpha}^1}{\partial III_\Theta}\bigg|_{\underline{\Omega}} \\
\tilde{b}_{21} &= \frac{\partial\, {}^m\tilde{\alpha}^2}{\partial I_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{22} = \frac{\partial\, {}^m\tilde{\alpha}^2}{\partial II_\Theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{23} = \frac{\partial\, {}^m\tilde{\alpha}^2}{\partial III_\Theta}\bigg|_{\underline{\Omega}} \\
\tilde{b}_{31} &= \frac{\partial\, {}^m\tilde{\alpha}^0}{\partial \theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{32} = \frac{\partial\, {}^m\tilde{\alpha}^1}{\partial \theta}\bigg|_{\underline{\Omega}}, \quad \tilde{b}_{33} = \frac{\partial\, {}^m\tilde{\alpha}^2}{\partial \theta}\bigg|_{\underline{\Omega}}
\end{aligned}
\tag{3.236}
$$

We can write (3.233) as

$$
\begin{aligned}
[m] = \tilde{b}_0[I] &+ \tilde{b}_1[{}^\Theta_s J] + \tilde{b}_2[{}^\Theta_s J]^2 \\
&+ \tilde{b}_{01} I_\Theta[I] + \tilde{b}_{02} II_\Theta[I] + \tilde{b}_{03} III_\Theta[I] \\
&+ \tilde{b}_{11} I_\Theta[{}^\Theta_s J] + \tilde{b}_{12} II_\Theta[{}^\Theta_s J] + \tilde{b}_{13} III_\Theta[{}^\Theta_s J] \\
&+ \tilde{b}_{21} I_\Theta[{}^\Theta_s J]^2 + \tilde{b}_{22} II_\Theta[{}^\Theta_s J]^2 + \tilde{b}_{23} III_\Theta[{}^\Theta_s J]^2 \\
&+ \tilde{b}_{31}\big(\theta - \theta_{\underline{\Omega}}\big)[I] + \tilde{b}_{32}\big(\theta - \theta_{\underline{\Omega}}\big)[{}^\Theta_s J] + \tilde{b}_{33}\big(\theta - \theta_{\underline{\Omega}}\big)[{}^\Theta_s J]^2
\end{aligned}
\tag{3.237}
$$

$\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$; $\tilde{b}_{ij}$; $i = 0,1,2,3$; $j = 1,2,3$ are material coefficients defined in the known configuration $\underline{\Omega}$. These are functions of the invariants of $[{}^\Theta_s J]$ and θ in $\underline{\Omega}$.

The constitutive theory for $[m]$ defined by (3.237) is based on integrity, hence is complete. It requires fifteen material coefficients and $[m]$ in (3.237) is up to a fifth degree polynomial in the components of $[{}^\Theta_s J]$ or rotation gradients, but is linear in temperature θ. Simplified forms of the constitutive theory (3.237) will be considered in a later section.

3.5.7 Constitutive theory for $[m]$ using strain energy density function ${}^m\pi$ (Approach II)

If ${}^m\pi$ is the strain energy density function due to the conjugate pair ($[m]$, $[{}_s^\Theta J]$), then following the details in section 3.5.2 for the work conjugate pair ($[{}_s\sigma], [\varepsilon]$) we can derive the following (similar to equation (3.192)) for the moment tensor $[m]$. or

$$[m] = [m]^T = \rho_0 \frac{\partial({}^m\pi)}{\partial[{}_s^\Theta J]} \tag{3.238}$$

Choosing

$${}^m\pi =^m \pi \left(I_\Theta, II_\Theta, III_\Theta, \theta\right) \tag{3.239}$$

and using in (3.238) and following derivation parallel to section 3.5.2 we can derive

$$[m] = {}^m\underset{\sim}{\alpha}^0[I] + {}^m\underset{\sim}{\alpha}^1[{}_s^\Theta J] + {}^m\underset{\sim}{\alpha}^{-1}[{}_s^\Theta J]^{-1} \tag{3.240}$$

Using Hamilton-Cayley theorem [71] to substitute for $[{}_s^\Theta J]^{-1}$ we obtain

$$[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}_s^\Theta J] + {}^m\tilde{\alpha}^2[{}_s^\Theta J]^2 \tag{3.241}$$

Since ${}^\sigma\underset{\sim}{\alpha}^i = {}^\sigma\underset{\sim}{\alpha}^i(I_\Theta, II_\Theta, III_\Theta, \theta)$; $i = 0, 1, 2$ we can conclude from (3.210) that

$${}^\sigma\tilde{\alpha}^i = {}^\sigma\tilde{\alpha}^i(I_\Theta, II_\Theta, III_\Theta, \theta);\ i = 0, 1, 2 \tag{3.242}$$

This constitutive theory is the same as the one derived using the theory of generators and invariants (section 3.5.6), thus determination of the material coefficients follows the same procedure as used in section 3.5.6 and finally we obtain exactly the same constitutive theory with the same definition of material coefficients (equation 3.237).

3.5.8 Constitutive theory for $[{}_s^\Theta J]$ in terms of $[m]$ based on complementary strain energy density function ${}^m\pi^c$

If ${}^m\pi^c$ is the complementary strain energy density function due to conjugate pair ($[m], [{}_s^\Theta J]$), then following the details in section 3.5.3 for conjugate pair ($[{}_s\sigma], [\varepsilon]$) we can derive the following (similar to equation (3.212)) for $[{}_s^\Theta J]$ (Also see reference [71] for more details on the method).

$$[{}_s^\Theta J] = \rho_0 \frac{\partial({}^m\pi^c)}{\partial[m]} \tag{3.243}$$

In which we assume

$${}^m\pi^c = {}^m\pi^c \left(I_m, II_m, III_m, \theta\right) \tag{3.244}$$

I_m, II_m, III_m are the principal invariants of the moment tensor $[m]$. Substituting (3.244) in (3.243) and following the derivation parallel to section 3.5.3 we can derive

$$[{}_s^\Theta J] = {}^J\alpha^0[I] + {}^J\alpha^1[m] + {}^J\alpha^{-1}[m]^{-1} \tag{3.245}$$

Using the Hamilton-Cayley theorem to substitute for $[m]^{-1}$ we obtain

$$[{}_s^{\Theta}J] = {}^J\tilde{\alpha}^0[I] + {}^J\tilde{\alpha}^1[m] + {}^J\tilde{\alpha}^2[m]^2 \tag{3.246}$$

in which

$${}^J\tilde{\alpha}^i = {}^J\tilde{\alpha}^i\left(I_m, II_m, III_m, \theta\right);\ i = 0,1,2 \tag{3.247}$$

I_m, II_m, III_m are the principal invariants of the tensor $[m]$. Clearly ${}^J\tilde{\alpha}^i$; $i = 0,1,2$ are functions of ${}^J\alpha^i$; $i = -1,0,1$ and ${}^J\alpha^i$ are functions of I_m, II_m, III_m, and θ, (3.247) is valid. Material coefficients in (3.246) are derived using exactly the same approach based on Taylor series as used in section 3.5.1, hence is not repeated here for the sake of brevity.

3.5.9 Constitutive theory for $[m]$ using strain energy density function ${}^m\pi([{}_s^{\Theta}J],\theta)$ and expanding it in Taylor series about a known configuration $\underline{\Omega}$

Consider ${}^m\pi = {}^m\pi([{}_s^{\Theta}J],\theta)$ and expand ${}^m\pi$ in $[{}_s^{\Theta}J]$ about a known configuration $\underline{\Omega}$ using Taylor series.

$$\begin{aligned}
{}^m\pi = {}^m\pi\big|_{\underline{\Omega}} &+ \frac{\partial({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right) \\
&+ \frac{1}{2!}\left.\frac{\partial^2({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})\partial({}_s^{\Theta}J_{kl})}\right|_{\underline{\Omega}}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{kl} - ({}_s^{\Theta}J_{kl})_{\underline{\Omega}}\right) \\
&+ \frac{1}{3!}\left.\frac{\partial^3({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})\partial({}_s^{\Theta}J_{kl})\partial({}_s^{\Theta}J_{pq})}\right|_{\underline{\Omega}}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{kl} - ({}_s^{\Theta}J_{kl})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{pq} - ({}_s^{\Theta}J_{pq})_{\underline{\Omega}}\right) + \cdots
\end{aligned} \tag{3.248}$$

Let

$$\begin{aligned}
{}^m\pi\big|_{\underline{\Omega}} &= {}^mC \\
\left.\frac{\partial({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})}\right|_{\underline{\Omega}} &= {}^mC_{ij} \\
\left.\frac{\partial^2({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})\partial({}_s^{\Theta}J_{kl})}\right|_{\underline{\Omega}} &= {}^m\hat{C}_{ijkl} \\
\left.\frac{\partial^3({}^m\pi)}{\partial({}_s^{\Theta}J_{ij})\partial({}_s^{\Theta}J_{kl})\partial({}_s^{\Theta}J_{pq})}\right|_{\underline{\Omega}} &= {}^m\tilde{C}_{ijklpq}
\end{aligned} \tag{3.249}$$

Substituting from (3.249) into (3.248)

$$\begin{aligned}
{}^m\pi = {}^mC &+ {}^mC_{ij}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right) + {}^m\hat{C}_{ijkl}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{kl} - ({}_s^{\Theta}J_{kl})_{\underline{\Omega}}\right) \\
&+ {}^m\tilde{C}_{ijklpq}\left({}_s^{\Theta}J_{ij} - ({}_s^{\Theta}J_{ij})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{kl} - ({}_s^{\Theta}J_{kl})_{\underline{\Omega}}\right)\left({}_s^{\Theta}J_{pq} - ({}_s^{\Theta}J_{pq})_{\underline{\Omega}}\right) + \cdots
\end{aligned} \tag{3.250}$$

Substituting ${}^m\pi$ from (3.250) into (3.238) and differentiating ${}^m\pi$ with respect to $[{}_s^\Theta J]$ and noting that partial derivatives of (3.249) with respect to $[{}_s^\Theta J]$ are zero and that

$$
\begin{aligned}
&\frac{\partial}{\partial({}_s^\Theta J_{mn})}\left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right) = \delta_{im}\delta_{jn} \\
&\frac{\partial}{\partial({}_s^\Theta J_{mn})}\left(\left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right)\left({}_s^\Theta J_{kl} - ({}_s^\Theta J_{kl})_{\underline{\Omega}}\right)\right) \\
&\qquad = \delta_{im}\delta_{jn}\left({}_s^\Theta J_{kl} - ({}_s^\Theta J_{kl})_{\underline{\Omega}}\right) + \left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln} \\
&\frac{\partial}{\partial({}_s^\Theta J_{mn})}\left(\left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right)\left({}_s^\Theta J_{kl} - ({}_s^\Theta J_{kl})_{\underline{\Omega}}\right)\left({}_s^\Theta J_{pq} - ({}_s^\Theta J_{pq})_{\underline{\Omega}}\right)\right) \\
&\qquad = \delta_{im}\delta_{jn}\left({}_s^\Theta J_{kl} - ({}_s^\Theta J_{kl})_{\underline{\Omega}}\right)\left({}_s^\Theta J_{pq} - ({}_s^\Theta J_{pq})_{\underline{\Omega}}\right) \\
&\qquad + \left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln}\left({}_s^\Theta J_{pq} - ({}_s^\Theta J_{pq})_{\underline{\Omega}}\right) \\
&\qquad + \left({}_s^\Theta J_{ij} - ({}_s^\Theta J_{ij})_{\underline{\Omega}}\right)\left({}_s^\Theta J_{kl} - ({}_s^\Theta J_{kl})_{\underline{\Omega}}\right)\delta_{pm}\delta_{qn}
\end{aligned}
\tag{3.251}
$$

We obtain the following (note that ρ_0 is absorbed in the coefficients in (3.252)).

$$
m_{mn} = (m_{mn})_{\underline{\Omega}} + C_{mnij}{}_s^\Theta J_{ij} + \bar{\bar{C}}_{mnijkl}{}_s^\Theta J_{ij}{}_s^\Theta J_{kl} + \cdots \tag{3.252}
$$

In obtaining (3.252) we collect those terms that are defined in the known configuration $\underline{\Omega}$ to define coefficients of $[{}_s^\Theta J]$ and $[{}_s^\Theta J]^2$ and we use symmetry of the coefficients [71] i.e. $\hat{C}_{mnij} = \hat{C}_{ijmn} \ldots$ etc.

3.5.10 Constitutive theory of $[{}_s^\Theta J]$ using complementary strain energy density function ${}^m\pi^c([m], \theta)$ and expanding it in Taylor series about a known configuration (Approach II)

Considering ${}^m\pi^c ={}^m \pi^c([m], \theta)$ and expanding this in Taylor series in $[m]$ about a known configuration $\underline{\Omega}$, then using (3.243) and following exactly the same procedure as in section 3.5.9 we can derive a constitutive theory for $[{}_s^\Theta J]$ as a function of $[m]$ that is exactly parallel to (3.232). Details are straight forward and hence omitted for the sake of brevity.

3.6 Remarks on the constitutive theories (Approach II)

In section 3.5.1 through 3.5.10 the most general constitutive theories have been derived using approach II for $[{}_s\sigma]$ and $[m]$ using the theory of generators and invariants and strain energy density functions ${}^{s\sigma}\pi$ and ${}^m\pi$. Additionally, the constitutive theories for $[{}_s\sigma]$ and $[m]$ are also presented using Taylor series expansions and the strain energy density functions. Constitutive theories for $[\varepsilon]$ and $[{}_s^\Theta J]$ in terms of $[{}_s\sigma]$ and $[m]$ have also been derived using the theory of generators and invariants and the complementary strain energy density functions ${}^{s\sigma}\pi^c$ and ${}^m\pi^c$ including the constitutive theories based on Taylor series expansions and the complementary strain energy density functions. In the following we make some specific remarks pertaining to the specific constitutive theories presented so far.

1. The constitutive theory for $[{}_s\sigma]$ resulting from the theory of generators and invariants and the strain energy density function ${}^{s\sigma}\pi$ are the same. Thus, when considering simplified theories

we can consider either one. The same is true for the constitutive theories for $[m]$ derived using the theory of generators and invariants and the strain energy density function ${}^m\pi$.

2. The constitutive theories derived for $[{}_s\sigma]$ and $[m]$ using Taylor series expansions violate the frame invariance principle as the material coefficients are functions of the argument tensors, not their invariants (in a known configuration). Unfortunately this is a common and serious drawback of the approaches for deriving constitutive theories that are based on Taylor series expansions.

3. The constitutive theory for $[\varepsilon]$ in terms of $[{}_s\sigma]$ resulting from the theory of generators and invariants and the complementary strain energy density functions ${}^{s\sigma}\pi^c$ are also the same. The same is true for the constitutive theory for $[{}^{\Theta}_{s}J]$ in terms of $[m]$ derived using the theory of generators and invariants and the complementary strain energy density function ${}^{s\sigma}\pi^c$.

4. The constitutive theories for $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ in terms of $[{}_s\sigma]$ and $[m]$ derived using the complementary strain energy density functions ${}^{s\sigma}\pi^c$ and ${}^m\pi^c$ and the Taylor series expansions also violate the frame invariance principle as the material coefficients in these theories are functions of the argument tensors and not of their invariants as required by the axioms of the constitutive theory.

3.7 Simplified form of the constitutive theories for $[{}_s\sigma]$ and $[m]$

The constitutive theories derived in sections 3.5.1 through 3.5.10 when based on integrity contain too many material coefficients. Simplified forms of these constitutive theories containing fewer material coefficients are necessary for determination of material coefficients as well as for their use in practical applications. In this section we consider some simplified forms of these theories. Based on the remarks in section 3.6, we only consider the constitutive theories derived using the theory of generators and invariants.

3.7.1 Simplified constitutive theory for $[{}_s\sigma]$

Using the most general form of the constitutive theory for $[{}_s\sigma]$ given by (3.182), we can derive various simplified constitutive theories for $[{}_s\sigma]$. For example, if we limit the constitutive theory for $[{}_s\sigma]$ to only up to second degree terms in the components of $[\varepsilon]$, then we obtain the following.

$$\begin{aligned}
[{}_s\sigma] = {} & b_0[I] + b_1[\varepsilon] + b_2[\varepsilon]^2 \\
& + b_{01} I_\varepsilon [I] + b_{02} II_\varepsilon [I] + b_{11} I_\varepsilon [\varepsilon] \\
& + b_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + b_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] + b_{33}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon]^2
\end{aligned} \tag{3.253}$$

A further simplification of (3.253) would be a constitutive theory for $[{}_s\sigma]$ that is linear in the components of $[\varepsilon]$.

$$[{}_s\sigma] = b_0[I] + b_1[\varepsilon] + b_{01} I_\varepsilon [I] + b_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + b_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] \tag{3.254}$$

If we redefine material coefficients in (3.254), we can write

$$[{}_s\sigma] = (\sigma_0)_{\underline{\Omega}}[I] + 2\tilde{\mu}_{\underline{\Omega}}[\varepsilon] + \tilde{\lambda}_{\underline{\Omega}}(\mathrm{tr}[\varepsilon])[I] - ({}^1\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + ({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] \tag{3.255}$$

If we neglect $\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon]$ terms in (3.255), then we obtain

$$[{}_s\sigma] = (\sigma_0)_{\underline{\Omega}}[I] + 2\tilde{\mu}_{\underline{\Omega}}[\varepsilon] + \tilde{\lambda}_{\underline{\Omega}}(\mathrm{tr}[\varepsilon])[I] - (\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[I] \tag{3.256}$$

This is the simplest possible constitutive theory for $[{}_s\sigma]$. $[{}_s\sigma]$ in (3.256) can also be represented in matrix and vector notation (Voigt's notation). See reference [71] for details. In (3.255) $(\sigma_0)_{\underline{\Omega}} = b_0$, $2\tilde{\mu}_{\underline{\Omega}} = b_1$, $\tilde{\lambda}_{\underline{\Omega}} = b_{01}$, $({}^1\tilde{\alpha}_{tm})_{\underline{\Omega}} = b_{31}$ and $({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}} = b_{32}$.

3.7.2 Simplified constitutive theory for $[m]$

Using the most general form of the constitutive theory for $[m]$ given by (3.237), we can derive various simplified constitutive theories for $[m]$. For example, if we limit the constitutive theory for $[m]$ to only up to second degree terms in the components of $[{}^\Theta_s J]$, then we obtain the following.

$$\begin{aligned}[m] = {} & \tilde{b}_0[I] + \tilde{b}_1[{}^\Theta_s J] + \tilde{b}_2[{}^\Theta_s J]^2 \\ & + \tilde{b}_{01} I_J [I] + \tilde{b}_{02} II_J [I] + \tilde{b}_{11} I_J [{}^\Theta_s J] \\ & + \tilde{b}_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + \tilde{b}_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[{}^\Theta_s J] + \tilde{b}_{33}\left(\theta - \theta_{\underline{\Omega}}\right)[{}^\Theta_s J]^2\end{aligned} \tag{3.257}$$

A further simplification of (3.257) would be a constitutive theory for $[m]$ that is linear in the components of $[{}^\Theta_s J]$.

$$[m] = \tilde{b}_0[I] + \tilde{b}_1[{}^\Theta_s J] + \tilde{b}_{01} I_J [I] + \tilde{b}_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + \tilde{b}_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[{}^\Theta_s J] \tag{3.258}$$

If we neglect $\left(\theta - \theta_{\underline{\Omega}}\right)[{}^\Theta_s J]$ terms in (3.258), then we obtain

$$[m] = (m_0)_{\underline{\Omega}}[I] + 2\,{}^m\tilde{\mu}_{\underline{\Omega}}[{}^\Theta_s J] + {}^m\tilde{\lambda}_{\underline{\Omega}}(\mathrm{tr}[{}^\Theta_s J])[I] - {}^m\tilde{\alpha}_{tm}\left(\theta - \theta_{\underline{\Omega}}\right)[I] \tag{3.259}$$

This is the simplest possible constitutive theory for $[m]$. In (3.259) $(m_0)_{\underline{\Omega}} = \tilde{b}_0$, $2\,{}^m\tilde{\mu}_{\underline{\Omega}} = \tilde{b}_1$, ${}^m\tilde{\lambda}_{\underline{\Omega}} = \tilde{b}_{01}$, ${}^m\tilde{\alpha}_{tm} = \tilde{b}_{31}$ and $({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}} = \tilde{b}_{32}$. $[{}_s\sigma]$ in (3.259) can also be represented in matrix and vector notation (Voigt's notation). See reference [71] for details.

3.8 Constitutive theory for heat vector $\boldsymbol{q}$

The constitutive theory for $\boldsymbol{q}$ can be derived: (i) using $\boldsymbol{q} = \boldsymbol{q}([{}^d_s J], [{}^\Theta_s J], \{g\}, \theta)$ in (3.124) and by using the theory of generators and invariants (ii) or using $\boldsymbol{q} \cdot \boldsymbol{g} \leq 0$ (inequality (3.130)) resulting from the entropy inequality.

3.8.1 Constitutive theory for $\boldsymbol{q}$ using the theory of generators and invariants

Consider $\boldsymbol{q}$ in (3.124)

$$\boldsymbol{q} = \boldsymbol{q}\left({}_s^d\boldsymbol{J}, {}_s^\Theta\boldsymbol{J}, \boldsymbol{g}, \theta\right) \tag{3.260}$$

Let $\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$; $i = 1, 2, \dots, \tilde{N}$ be the combined generators of the argument tensors $[{}_s^d J]$, $[{}_s^\Theta J]$, and $\{g\}$ that are tensors of rank one. Let ${}^q\underset{\sim}{I}^j$; $j = 1, 2, \dots, \tilde{M}$ be the combined invariants of the same argument tensors. Then, we can express $\{q\}$ as a linear combination of $\left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\}$; $i = 1, 2, \dots, \tilde{N}$.

$$\{q\} = -\sum_{i=1}^{\tilde{N}} {}^q\underset{\sim}{\alpha}^i \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{3.261}$$

The absence of a unit vector in (3.261) is due to the fact that a uniform temperature field does not contribute to $\{q\}$. The negative sign in (3.261) is because a positive $\{q\}$ in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter. The coefficients ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ are functions of ${}^q\underset{\sim}{I}^j$; $j = 1, 2, \dots, \tilde{M}$ and θ in the current configuration. To determine the material coefficients from ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ (in the current configuration), we consider Taylor series expansion of each ${}^q\underset{\sim}{\alpha}^i$; $i = 1, 2, \dots, \tilde{N}$ about a known configuration $\underline{\Omega}$ in θ and ${}^q\underset{\sim}{I}^j$; $j = 1, 2, \dots, \tilde{M}$ and retain only up to linear terms in θ and the invariants.

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right);\; i = 1, 2, \dots, \tilde{N} \tag{3.262}$$

${}^q\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}}$, $\left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$, and $\left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}$ are functions of $\theta\big|_{\underline{\Omega}}$ and ${}^q\underset{\sim}{I}^j\big|_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$ whereas ${}^q\underset{\sim}{\alpha}^i$ are functions of the same quantities in the current configuration. By substituting (3.262) in (3.261) we obtain the most general form of the constitutive theory for $\boldsymbol{q}$ that is based on integrity i.e. complete basis. Details are given in the following.

$$\{q\} = -\sum_{i=1}^{\tilde{N}} \left({}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \left.\frac{\partial {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right) \right) \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{3.263}$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (3.263) that are defined in the current configuration i.e. coefficients of $\left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\}$, ${}^q\underset{\sim}{I}^j\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$ and $(\theta-\theta_{\underline{\Omega}})\left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\}$ and defining the following

$$\begin{aligned}
{}^q b_i &= {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{\tilde{M}} \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^q c_{ij} &= \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \\
{}^q d_i &= \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial \theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{3.264}$$

for $i = 1, 2, \dots, \tilde{N}$ and $j = 1, 2, \dots, \tilde{M}$ and using these in (3.263) we can write

$$\{q\} = -\sum_{i=1}^{\tilde{N}} {}^q b_i \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}}\sum_{i=1}^{\tilde{M}} {}^q c_{ij}\, {}^q\underset{\sim}{I}^j \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}} {}^q d_i(\theta-\theta_{\underline{\Omega}}) \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{3.265}$$

${}^q b_i$, ${}^q c_{ij}$, and ${}^q d_i$ are material coefficients defined in a known configuration $\underline{\Omega}$. The constitutive theory for $\boldsymbol{q}$ defined by(3.265) requires $(\tilde{N}+\tilde{N}\tilde{M}+\tilde{N})$ material coefficients. The material coefficients are functions of $\theta|_{\underline{\Omega}}$ and $({}^q\underset{\sim}{I}^j)_{\underline{\Omega}}$; $j = 1, 2, \dots, \tilde{M}$. This constitutive theory for $\boldsymbol{q}$ is based on integrity, hence is complete.

3.8.2 Simplified constitutive theory for heat vector $\boldsymbol{q}$

Much simpler (but with limitations) constitutive theories for $\boldsymbol{q}$ can be derived if we limit its argument tensors. Consider

$$\boldsymbol{q} = \boldsymbol{q}\,(\boldsymbol{g}, \theta) \tag{3.266}$$

In this case, we have only one generator and one invariant (i.e. $\tilde{N} = 1$ and $\tilde{M} = 1$).

$$\left\{{}^q\underset{\sim}{\boldsymbol{G}}^1\right\} = \{q\}\,; \qquad {}^q\underset{\sim}{I}^1 = \{g\}^T\{g\} \tag{3.267}$$

Following the general derivation in section 3.8.1 we can write the following for $\tilde{N} = 1$, $\tilde{M} = 1$

$$\{q\} = -{}^q b_1\{g\} - {}^q c_{11}\left(\{g\}^T\{g\}\right)\{g\} - {}^q d_1(\theta-\theta_{\underline{\Omega}})\{g\} \tag{3.268}$$

Material coefficients in (3.268) are defined by (3.264). This constitutive theory is cubic in $\{g\}$, requires only three material coefficients and is the most general constitutive theory based on (3.266). If we denote ${}^q b_1 = k_1|_{\underline{\Omega}}$ and ${}^q c_{11} = k_2|_{\underline{\Omega}}$, then (3.268) can be written as

$$\{q\} = -\,k_1|_{\underline{\Omega}}\{g\} - k_2|_{\underline{\Omega}}\left(\{g\}^T\{g\}\right)\{g\} - {}^q d_1(\theta-\theta_{\underline{\Omega}})\{g\} \tag{3.269}$$

If we neglect the $(\theta - \theta_{\underline{\Omega}})$ term in (3.269), then we obtain

$$\{q\} = -\left. k_1\right|_{\underline{\Omega}} \{g\} - \left. k_2\right|_{\underline{\Omega}} \left(\{g\}^T \{g\}\right) \{g\} \tag{3.270}$$

If we assume that $\{q\}$ is a linear function of $\{g\}$, then we have

$$\{q\} = -\left. k_1\right|_{\underline{\Omega}} \{g\} \tag{3.271}$$

Equation (3.271) is the Fourier heat conduction law in which the thermal conductivity $\left. k_1\right|_{\underline{\Omega}}$ can be a function of $\left.\theta\right|_{\underline{\Omega}}$ and $\left(\{g\}^T \{g\}\right)_{\underline{\Omega}}$.

3.8.3 Constitutive theory for $\boldsymbol{q}$ *based on conditions resulting from the entropy inequality*

We recall that satisfying the entropy inequality requires that

$$\{q\}^T \{g\} \leq 0 \tag{3.272}$$

must hold. The derivation of the constitutive theory for $\{q\}$ based on (3.272) is standard and can be found in reference [71] and many others. The resulting constitutive theory for $\{q\}$ can be written as (3.271) except that in this case $\left. k_1\right|_{\underline{\Omega}} = k_1(\theta_{\underline{\Omega}})$ i.e. the conductivity can only be a function of temperature $\left.\theta\right|_{\underline{\Omega}}$ and not the temperature $\left.\theta\right|_{\underline{\Omega}}$ and the invariant $\left(\{g\}^T \{g\}\right)_{\underline{\Omega}}$ as there is no basis for dependence of $\left. k_i\right|_{\underline{\Omega}}$ on $\left(\{g\}^T \{g\}\right)_{\underline{\Omega}}$.

3.9 Closure of mathematical model and comments on the constitutive theories

In this mathematical model, the dependent variables are (numbers in the lower case brackets indicate the number of variables):

$$\begin{gathered} v_i(3)\ , \quad {}_s\boldsymbol{\sigma}(6)\ , \quad {}_a\boldsymbol{\sigma}(3)\ , \quad \boldsymbol{m}(6)\ , \quad e(1)\ , \quad \boldsymbol{q}(3) \\ \Phi(1)\ , \quad \eta(1)\ , \quad \theta(1) \quad ; \quad \text{a total of } 25 \end{gathered} \tag{3.273}$$

In these, Φ and η will be eliminated from the list of variables. The specific internal energy is a function of ρ and θ, that is $e(\rho, \theta)$ for most general case of compressible matter, hence e is also eliminated from the list of dependent variables. This leaves us with remaining 22 dependent variables in the mathematical model. We have linear momentum equation (3), angular momentum equation (3), energy equation (1) and, from entropy inequality we have constitutive theories for ${}_s\boldsymbol{\sigma}$ (6), $\boldsymbol{m}$ (6) and $\boldsymbol{q}$ (3), a total of 22 equations, hence this mathematical model will have closure once we have constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$ and $\boldsymbol{q}$. Development of the constitutive theories is clearly treatment of matter specific physics. This mathematical model is suited for solid matter experiencing small to moderate deformation both compressible and incompressible.

4. MODEL PROBLEMS AND THEIR SOLUTIONS*

4.1 Model problems and solutions for internal polar fluent continua

In this section we consider three model problems describing boundary value problems for internal polar thermofluids considered in this paper. The first model problem consists of fully developed flow of a non-isothermal internal polar thermofluid between parallel plates. The second model problem consists of a lid driven square cavity using isothermal internal polar thermofluid. The third model problem consists of flow past a sudden expansion using an isothermal internal polar thermofluid. The objective in these model problems is to investigate the influence of internal polar physics on the solutions when compared with the non-polar case. All model problems are boundary value problems, sufficient to illustrate the main features of the internal polar physics and associated theories.

For the first model problem, fully developed flow between parallel plates, the physics appears rather simple and the resulting mathematical models for internally non-polar continua easily permits analytical or theoretical solutions, however when using internal polar continuum theories the resulting mathematical models are complex enough not to permit complete theoretical or analytical solutions, hence we consider numerical solutions using finite element method in which the local approximations are considered in higher order scalar product spaces (*hpk* framework) and the resulting integral forms are variationally consistent, thus unconditional stability of the computations is ensured. Furthermore in all computations the order k of the approximation space is chosen so that all integrals are over discretizations are Riemann. For this choice of k, when the integrated sum of squares of the residuals approaches zero we are ensured that the governing differential equations in the mathematical models are satisfied in the pointwise sense. Hence, such computed solutions have the same accuracy up to certain order derivatives (depending on k) as the theoretical solutions. This is an essential and necessary feature of the computed solution so that the difference between the internal polar and non-polar theories can be accepted without doubting the accuracy of computations.

In the case of lid driven cavity and sudden expansion, the mathematical model incorporating polar physics also does not permit analytical solutions. Thus, in this case we use the same *hpk* framework and finite element method with variationally consistent integral forms for obtaining numerical solutions as in the case of fully developed flow between parallel plates. The finite element computational framework has the same attributes and features for both boundary value problems in terms of higher degree p-version hierarchical approximations with higher order global differentiability.

*Portions of the derivation of the numeric results for fluent continua presented here appear in the article "Ordered Rate Constitutive Theories for Internal Polar Thermofluids" by K.S. Surana, M. Powell, and J.N. Reddy *Int. J. of Math. Sci. & Engg. Appls. (IJMSEA)* Vol. 9, No. 3 pp. 51–116 (2015) ©Ascent Journals. Portions of the numeric results for solid continua presented here appear in "Constitutive Theories for Internal Polar Thermoelastic Solid Continua" by K.S. Surana, M. Powell, and J.N. Reddy *J. of Pure and Applied Mathematics: Advances and Applications* Vol. 14, No. 2 pp. 89–150 (2015) ©Scientific Advances Publishers

4.1.1 *Model problem 1: Fully developed non-isothermal flow of an incompressible internal polar thermofluid between parallel plates*

Figure 4.1 shows a schematic of the flow between parallel plates. For simplicity we consider x, y (or $1, 2$) to represent x_1, x_2. Locations along the y axis in figure 4.1 are in fact $\bar{y}$, current positions of fluid particles. The mathematical model in this case consists of $\bar{x}$-momentum equation, balance of angular momentum, energy equation, constitutive theory for the deviatoric part of the symmetric shear stress ${}_d({}_s\bar{\sigma}_{yx})$, constitutive theory for the moment tensor and the constitutive theory for the heat vector. Antisymmetric part of the shear stress naturally appears in the momentum equation and the balance of angular momentum, which also contains gradients of the moment tensor. In representing stress and moment tensor we have dropped basis dependency as suggested by the choice of constitutive theories used for them. We have the following mathematical model (x-momentum, balance of angular momentum, energy equation, and the constitutive equations) in which all quantities have their usual units or dimensions (indicated by $\hat{}$ (hat)). We note that the constitutive theories used for stress and moment tensors are linear in $[\bar{D}]$, and $[{}^{\Theta}D]$ and the constitutive theory for heat vector is simple Fourier heat conduction law.

$$\frac{\partial\big({}_d({}_s\hat{\bar{\sigma}}_{yx})\big)}{\partial\hat{\bar{y}}} + \frac{\partial({}_a\hat{\bar{\sigma}}_{yx})}{\partial\hat{\bar{y}}} - \frac{\partial\hat{\bar{p}}}{\partial\hat{\bar{x}}} = 0 \tag{4.1}$$

$$\frac{\partial\hat{\bar{m}}_{zy}}{\partial\hat{\bar{y}}} - 2({}_a\hat{\bar{\sigma}}_{yx}) = 0 \tag{4.2}$$

$$\frac{\partial\hat{\bar{q}}}{\partial\hat{\bar{y}}} - {}_d({}_s\hat{\bar{\sigma}}_{yx})\left(\frac{\partial\hat{\bar{v}}_1}{\partial\hat{\bar{y}}}\right) - \hat{\bar{m}}_{zy}\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right) = 0 \tag{4.3}$$

$${}_d({}_s\hat{\bar{\sigma}}_{yx}) = \hat{\eta}|_{\underline{\Omega}}\frac{\partial\hat{\bar{v}}_1}{\partial\hat{\bar{y}}} \tag{4.4}$$

$$\hat{\bar{m}}_{zy} = 2^m\hat{\eta}|_{\underline{\Omega}}\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right) \tag{4.5}$$

$$\hat{\bar{q}} = -\hat{k}|_{\underline{\Omega}}\frac{\partial\hat{\theta}}{\partial\hat{\bar{y}}} \tag{4.6}$$

$$\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right) = \frac{\partial^2\bar{v}_1}{\partial\bar{y}^2} \tag{4.7}$$

We substitute (4.4) and (4.5) in (4.3) to ensure that dissipation terms are always positive (but keep (4.4) and (4.5) as part of the mathematical model also as they appear in other equations too). We also substitute (4.6) in (4.3) to eliminate (4.6) from the mathematical model. The resulting equations are (assuming $\hat{k}|_{\underline{\Omega}}$ to be constant):

$$\frac{\partial\big({}_d({}_s\hat{\bar{\sigma}}_{yx})\big)}{\partial\hat{\bar{y}}} + \frac{\partial({}_a\hat{\bar{\sigma}}_{yx})}{\partial\hat{\bar{y}}} - \frac{\partial\hat{\bar{p}}}{\partial\hat{\bar{x}}} = 0 \tag{4.8}$$

$$\frac{\partial\hat{\bar{m}}_{zy}}{\partial\hat{\bar{y}}} - 2({}_a\hat{\bar{\sigma}}_{yx}) = 0 \tag{4.9}$$

$$\hat{k}|_{\underline{\Omega}}\frac{\partial^2\hat{\bar{\theta}}}{\partial\hat{\bar{y}}^2} + \hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial\hat{\bar{v}}_1}{\partial\hat{\bar{y}}}\right)^2 + 2{}^m\hat{\eta}|_{\underline{\Omega}}\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right)^2 = 0 \tag{4.10}$$

$${}_d({}_s\hat{\bar{\sigma}}_{yx}) = \hat{\eta}|_{\underline{\Omega}}\frac{\partial\hat{\bar{v}}_1}{\partial\hat{\bar{y}}} \tag{4.11}$$

$$\hat{\bar{m}}_{zy} = 2{}^m\hat{\eta}|_{\underline{\Omega}}\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right) \tag{4.12}$$

$$\left({}^t\hat{\bar{\Theta}}_{z,\bar{y}}\right) = \frac{\partial^2\bar{v}_1}{\partial\bar{y}^2} \tag{4.13}$$

Equations (4.8)–(4.12) are five PDEs in five variables ${}_d({}_s\hat{\bar{\sigma}}_{yx}), ({}_a\hat{\bar{\sigma}}_{yx}), \hat{\bar{m}}_{zy}, \hat{\bar{v}}_1$, and ${}^t\hat{\bar{\Theta}}_{z,\bar{y}}$, hence this mathematical model (4.8)–(4.12) has closure. The flow is assumed to be pressure driven, hence $\frac{\partial\hat{\bar{p}}}{\partial\hat{\bar{x}}}$ is known. We nondimensionalize (4.8)–(4.12) using reference quantities (with subscript zero) and dimensionless quantities (without ˆ (hat)).

Let $L_0, p_0, \tau_0, \eta_0, k_0, m_0, \theta_0, v_0$ be reference length, pressure, stress, viscosity, conductivity, moment, temperature, velocity, etc., then the dimensionless quantities (without hat) are

$$\bar{y} = \frac{\hat{\bar{y}}}{L_0}, \quad \bar{\theta} = \frac{\hat{\bar{\theta}}}{\theta_0}, \quad \eta = \frac{\hat{\eta}|_{\underline{\Omega}}}{\eta_0}, \quad {}^m\eta = \frac{{}^m\hat{\eta}|_{\underline{\Omega}}}{\eta_0}$$

and assuming $p_0 = \tau_0 = \rho_0 v_0^2$, characteristic kinetic energy, we have

$$\bar{p} = \frac{\hat{\bar{p}}}{p_0}, \quad {}_d({}_s\bar{\sigma}_{yx}) = \frac{{}_d({}_s\hat{\bar{\sigma}}_{yx})}{\tau_0}, \quad ({}_a\bar{\sigma}_{yx}) = \frac{({}_a\hat{\bar{\sigma}}_{yx})}{\tau_0}$$

$$\bar{v}_1 = \frac{\hat{\bar{v}}_1}{v_0}, \quad \bar{m}_{zy} = \frac{\hat{\bar{m}}_{zy}}{m_0}, \quad k = \frac{\hat{k}|_{\underline{\Omega}}}{k_0}$$

Using these and substituting for quantities with a ˆ (hat) in terms of those without hat and reference quantities in (4.8)–(4.12), we obtain:

$$\frac{\partial({}_d({}_s\bar{\sigma}_{yx}))}{\partial\bar{y}} + \frac{\partial({}_a\bar{\sigma}_{yx})}{\partial\bar{y}} - \left(\frac{p_0}{\tau_0}\right)\frac{\partial\bar{p}}{\partial\bar{x}} = 0 \tag{4.14}$$

$$\frac{\partial\bar{m}_{zy}}{\partial\bar{y}} - \left(\frac{\tau_0 L_0}{m_0}\right) 2({}_a\bar{\sigma}_{yx}) = 0 \tag{4.15}$$

$$\left(\frac{k}{Br}\right)\frac{\partial^2\bar{\theta}}{\partial\bar{y}^2} + \eta\left(\frac{\partial\bar{v}_1}{\partial\bar{y}}\right)^2 + {}^m\eta\frac{(L_0)^2}{8}\left(\frac{\partial^2\bar{v}_1}{\partial\bar{y}^2}\right)^2 = 0 \tag{4.16}$$

$${}_d({}_s\bar{\sigma}_{yx}) = \eta\left(\frac{\eta_0 v_0}{L_0\tau_0}\right)\left(\frac{\partial\bar{v}_1}{\partial\bar{y}}\right) \tag{4.17}$$

$$\bar{m}_{zy} = {}^m\eta\frac{1}{2}\left(\frac{\eta_0 v_0}{m_0 L_0^2}\right)\left(\frac{\partial^2\bar{v}_1}{\partial\bar{y}^2}\right) \tag{4.18}$$

in which Br is Brinkmann's number. We choose physical quantities and reference quantities in such a way that all multipliers appearing in (4.14)–(4.18) due to nondimensionalization become unity. Such dimensionless form of the equations is sufficient to compare the solutions with and without internal polar physics. Thus (4.14)–(4.18) reduce to the following:

$$\frac{\partial({}_d({}_s\bar{\sigma}_{yx}))}{\partial\bar{y}} + \frac{\partial({}_a\bar{\sigma}_{yx})}{\partial\bar{y}} - \frac{\partial\bar{p}}{\partial\bar{x}} = 0 \tag{4.19}$$

$$\frac{\partial\bar{m}_{zy}}{\partial\bar{y}} - 2({}_a\bar{\sigma}_{yx}) = 0 \tag{4.20}$$

$$k\frac{\partial^2\bar{\theta}}{\partial\bar{y}^2} + \eta\left(\frac{\partial\bar{v}_1}{\partial\bar{y}}\right)^2 + {}^m\eta\frac{1}{8}\left(\frac{\partial^2\bar{v}_1}{\partial\bar{y}^2}\right)^2 = 0 \tag{4.21}$$

$${}_d({}_s\bar{\sigma}_{yx}) = \eta\left(\frac{\partial\bar{v}_1}{\partial\bar{y}}\right) \tag{4.22}$$

$$\bar{m}_{zy} = {}^m\eta\frac{1}{2}\left(\frac{\partial^2\bar{v}_1}{\partial\bar{y}^2}\right) \tag{4.23}$$

Equations (4.19)–(4.23) are used to compute numerical solutions. The material coefficient η is dimensionless viscosity and has a value of one. The second material coefficient ${}^m\eta$ (dimensionless) appears in the constitutive theory for the Cauchy moment tensor and is a measure of the deviation from the non-polar theory. When ${}^m\eta$ is zero the fluid is non-polar. When ${}^m\eta > 0$, the progressively increasing values of ${}^m\eta$ represent progressively more pronounced influence of rotation rates and hence more pronounced deviation from the non-polar theory. From the mathematical model we observe that the balance of linear momentum in the $\bar{x}-$direction shows that the gradient of $\bar{\sigma}_{yx}$ $(= {}_d({}_s\bar{\sigma}_{yx}) +_a \bar{\sigma}_{yx})$ in the $\bar{y}-$direction is equal to the negative pressure gradient in the $\bar{x}-$direction i.e. $\frac{\partial\bar{\sigma}_{yx}}{\partial\bar{y}} = -\frac{\partial\bar{p}}{\partial\bar{x}}$. This suggests that regardless of the values of ${}^m\eta$, for this model problem the total shear stress $\bar{\sigma}_{yx}$ is linear across the plates. This is a rather important piece of information that can be used as a check on the validity of the computed solutions.

The spatial domain $0 \le \bar{y} \le 2$ is discretized using 6 three node p-version $C^i(\bar{\Omega}^e_y)$ higher order continuity elements [97–104]. Numerical results are computed for p-level of 11 for all elements of the discretization with k, the order of the approximation space equal to 2 i.e. using local approximations of class $C^1(\bar{\Omega}_y^e)$. The finite element formulation used is based on the residual functional i.e. least squares finite element method. For these choices of h, p, and k the integrated sum of squares of the residual is $O(10^{-15})$ or lower confirming that the GDEs are satisfied accurately in the pointwise sense as for this choice of k all integrals over discretization $\bar{\Omega}_y^T = \cup_e \bar{\Omega}_y^e$ of $\bar{\Omega}_y$ are Riemann. In all numerical calculations $\eta = 1$ (dimensionless Newtonian viscosity) and the dimensionless parameter ${}^m\eta$ controlling the influence of rotation rate and its gradients is varied between $0 \le {}^m\eta \le 2$. Clearly for ${}^m\eta = 0$, we have purely non-polar behavior i.e. non-polar thermofluid. For progressively increasing values of ${}^m\eta$ (for $\eta = 1$) the fluid behavior progressively deviates from non-polar behavior. We compute solutions for ${}^m\eta = 0.0, 0.1, 0.25, 0.5, 1.0$ and 2.0. All computations are performed for $\frac{\partial \bar{p}}{\partial \bar{x}} = -0.1$ and is kept fixed while ${}^m\eta$ is varied. We discuss the results in the following.

Graphs of axial velocity $\bar{v}_1$, deviatoric symmetric Cauchy stress ${}_d({}_s\bar{\sigma}_{yx})$, antisymmetric stress ${}_a\bar{\sigma}_{yx}$, and total stress $\bar{\sigma}_{yx}$ versus distance $\bar{y}$ are shown in figures 4.2–4.5 for the chosen values of ${}^m\eta$. Plots of temperature $\bar{\theta}$, Cauchy moment tensor component and $\frac{\partial^2 \bar{v}_1}{\partial \bar{y}^2}$ versus distance $\bar{y}$ are shown in figures 4.6–4.8 for different values of ${}^m\eta$. Figure 4.9 shows a plot of flow rate $\bar{Q} = \int_0^2 \bar{v}_1(\bar{y}) d\bar{y}$ (for unit depth perpendicular to the plane of the paper) as a function of ${}^m\eta$. From the velocity graphs in figure 4.2 we observe that for ${}^m\eta = 0.0$ the velocity profile is the same as for pure Newtonian non-polar fluid. Progressively increasing values of ${}^m\eta$ results in progressively increasing resistance to flow due to rates of rotation gradients, hence the peak velocity value at the center of the flow domain diminishes with corresponding reduction in the axial velocity in the remaining flow domain. We observe that for ${}^m\eta$ values larger than 2.0 it is possible to completely choke the flow for the fixed value of $\frac{\partial \bar{p}}{\partial \bar{x}} = -0.1$ used here. This behavior exists regardless of the value of $\frac{\partial \bar{p}}{\partial \bar{x}}$. For ${}^m\eta = 0.0$, ${}_d({}_s\bar{\sigma}_{yx})$ is a linear function of $\bar{y}$ (figure 4.3) while ${}_a\bar{\sigma}_{yx}$ is exactly zero in the entire domain (figure 4.4) as they should be for non-polar Newtonian fluids. On the other hand for ${}^m\eta = 2.0$, ${}_a\bar{\sigma}_{yx}$ is close to being a linear function of $\bar{y}$ while ${}_d({}_s\bar{\sigma}_{yx})$ approaches zero (figures 4.3, 4.4). For this case the rate of rotation gradient behavior is the dominant physics. As expected $\bar{\sigma}_{yx}$ (figure 4.5) remains a linear function of $\bar{y}$, regardless of the value of ${}^m\eta$, as seen from the balance of linear momentum in $\bar{x}-$direction. From figure 4.6 we note that temperature $\bar{\theta}$ due to dissipation is most pronounced for ${}^m\eta = 0.0$ and diminishes with increasing values of ${}^m\eta$. Reduction of velocity and its gradients are of course responsible for this behavior. Plots of moments versus $\bar{y}$ in figure 4.7 show progressively increasing magnitude of Cauchy moment with progressively increasing ${}^m\eta$. Progressively reducing flow rate for progressively increasing values of ${}^m\eta$ in figure 4.9 are in agreement with the results presented in figure 4.2 and others.

This study demonstrates the influence of the rates of rotations incorporated in the internal polar fluent continuum theory when compared to non-polar fluent continuum theory used currently for homogeneous isotropic fluent continua. Progressively increasing value of ${}^m\eta$ of course imply that fluids in which the influence of rotation rates and their gradients is becoming more dominant in the flow physics.

4.1.2 Model problem 2: A square lid driven cavity

In this model problem we consider isothermal flow of an incompressible thermoviscous polar fluid in a square lid driven cavity and compare these solutions with the non-polar case. The mathematical

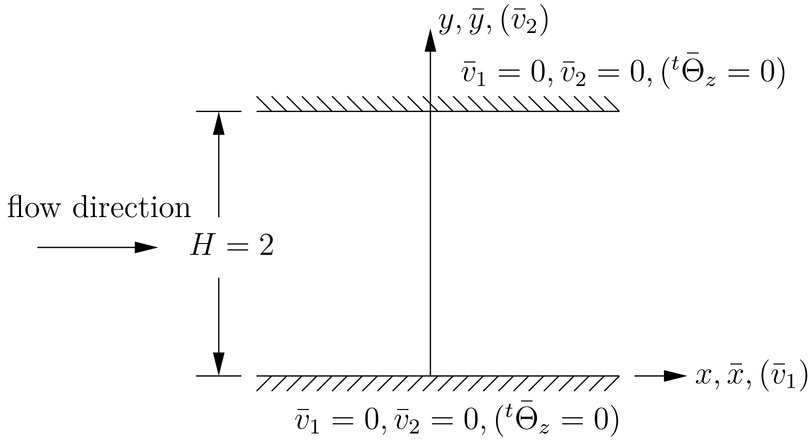


Figure 4.1: Fully developed non-isothermal flow of an incompressible polar thermofluid: schematic

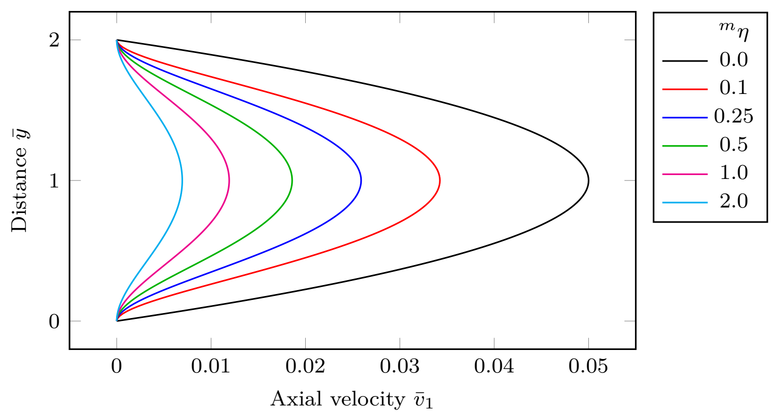


Figure 4.2: Axial velocity $\bar{v}_1$ versus distance $\bar{y}$

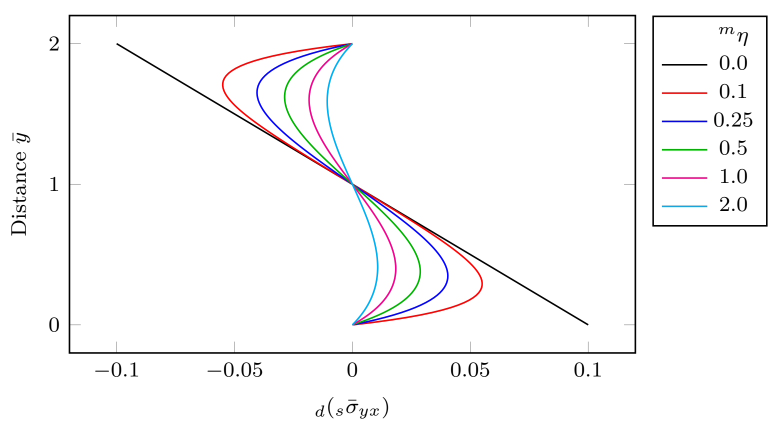


Figure 4.3: Deviatoric part of symmetric shear stress: ${}_d({}_s\bar{\sigma}_{yx})$ versus distance $\bar{y}$

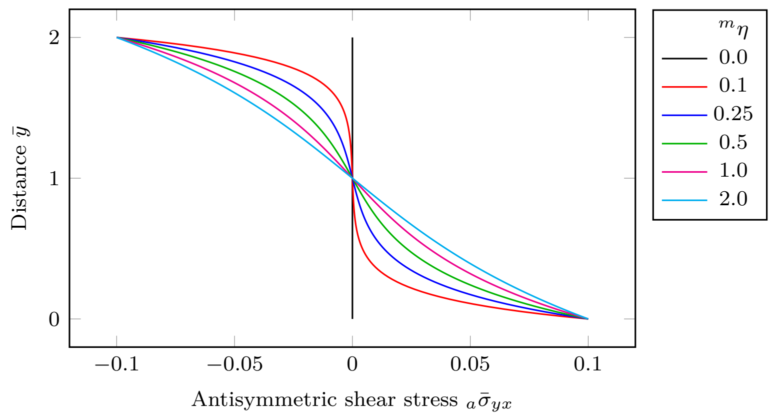


Figure 4.4: Antisymmetric shear stress ${}_a\bar{\sigma}_{yx}$ versus distance $\bar{y}$

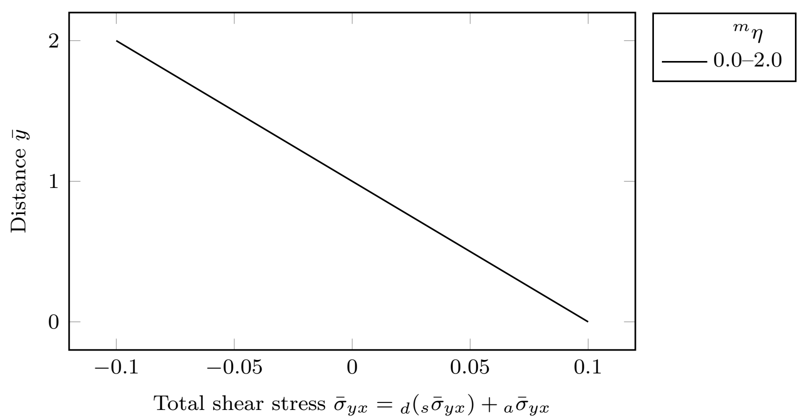


Figure 4.5: Total shear stress $\bar{\sigma}_{yx}$ versus distance $\bar{y}$

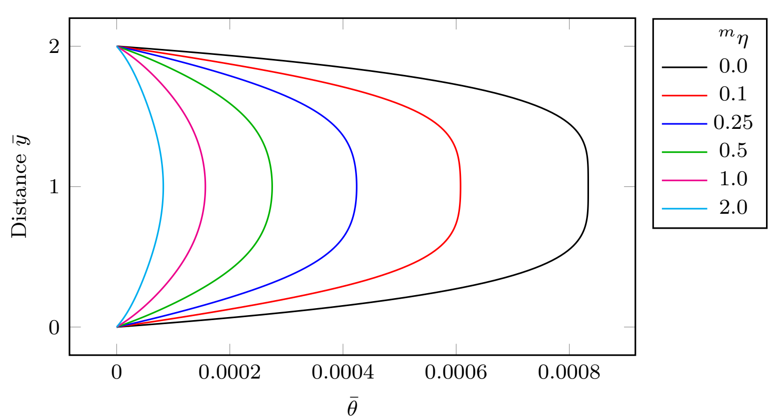


Figure 4.6: Temperature $\bar{\theta}$ versus distance $\bar{y}$

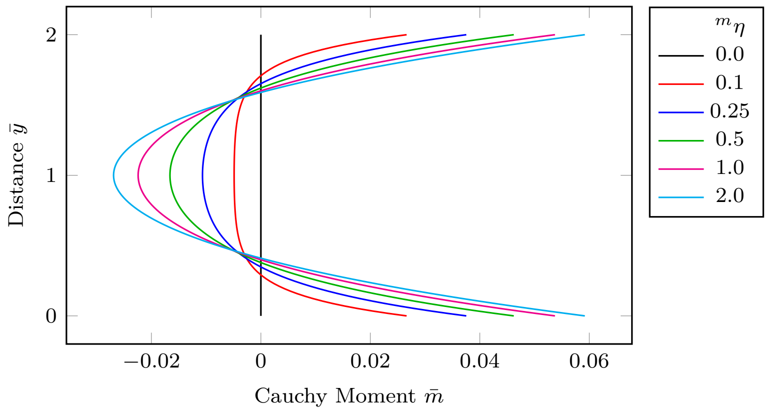


Figure 4.7: Cauchy moment $\bar{m}$ versus distance $\bar{y}$

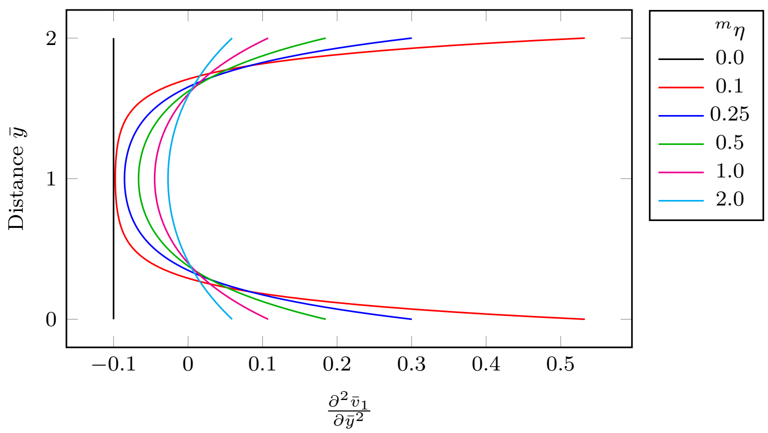


Figure 4.8: $\frac{\partial^2 \bar{v}_1}{\partial \bar{y}^2}$ versus distance $\bar{y}$

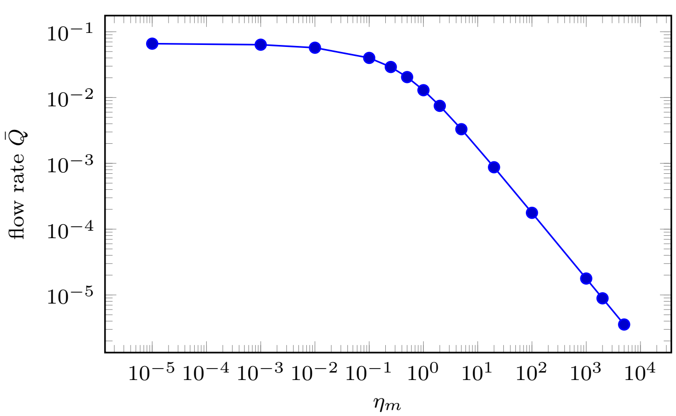


Figure 4.9: Flow rate versus $^m\eta$

model in this case consists of: continuity equation, x and y momentum equations, moment of moments and the constitutive theories for deviatoric part of the symmetric Cauchy stress tensor and the moment tensor. We recast these equations resulting from the conservation and balance laws as a system of first order partial differential equations. This has some advantages when computing numerical solutions using the finite element method in the hpk framework [97–104]. If we consider all quantities with usual dimensions indicated by ˆ (hat) on all of them, then we have the following mathematical model (using x, y, or $1, 2$) to represent x_1, x_2).

$$\hat{\bar{\rho}}\left(\frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{x}}} + \frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{y}}}\right) = 0 \tag{4.24}$$

$$\hat{\bar{\rho}}\left(\hat{\bar{v}}_1 \frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{x}}} + \hat{\bar{v}}_2 \frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{y}}}\right) + \frac{\partial \hat{\bar{p}}}{\partial \hat{\bar{x}}} - \left(\frac{\partial_d({}_s\bar{\sigma}_{xx})}{\partial \hat{\bar{x}}} + \frac{\partial_d({}_s\bar{\sigma}_{xy})}{\partial \hat{\bar{y}}} - \frac{\partial({}_a\bar{\sigma}_{xy})}{\partial \hat{\bar{y}}}\right) = 0 \tag{4.25}$$

$$\hat{\bar{\rho}}\left(\hat{\bar{v}}_1 \frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{x}}} + \hat{\bar{v}}_2 \frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{y}}}\right) + \frac{\partial \hat{\bar{p}}}{\partial \hat{\bar{y}}} - \left(\frac{\partial_d({}_s\bar{\sigma}_{xy})}{\partial \hat{\bar{x}}} + \frac{\partial_d({}_s\bar{\sigma}_{yy})}{\partial \hat{\bar{y}}} - \frac{\partial({}_a\bar{\sigma}_{xy})}{\partial \hat{\bar{x}}}\right) = 0 \tag{4.26}$$

$$\frac{\partial \hat{\bar{m}}_{zx}}{\partial \hat{\bar{x}}} + \frac{\partial \hat{\bar{m}}_{zy}}{\partial \hat{\bar{y}}} - 2({}_a\bar{\sigma}_{xy}) = 0 \tag{4.27}$$

$$\begin{aligned}
{}_d({}_s\hat{\bar{\sigma}}_{xx}) &= 2\hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{x}}}\right) \\
{}_d({}_s\hat{\bar{\sigma}}_{yy}) &= 2\hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{y}}}\right) \\
{}_d({}_s\hat{\bar{\sigma}}_{xy}) &= \hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{y}}} + \frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{x}}}\right)
\end{aligned} \tag{4.28}$$

$$\hat{\bar{m}}_{zx} = {}^m\hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial {}^t\hat{\bar{\Theta}}_z}{\partial \hat{\bar{x}}}\right) \tag{4.29}$$

$$\hat{\bar{m}}_{zy} = {}^m\hat{\eta}|_{\underline{\Omega}}\left(\frac{\partial {}^t\hat{\bar{\Theta}}_z}{\partial \hat{\bar{y}}}\right) \tag{4.30}$$

$${}^t\hat{\bar{\Theta}}_z = \frac{1}{2}\left(\frac{\partial \hat{\bar{v}}_1}{\partial \hat{\bar{y}}} + \frac{\partial \hat{\bar{v}}_2}{\partial \hat{\bar{x}}}\right) \tag{4.31}$$

Equations (4.24)–(4.31) are ten partial differential equations in ten variables: $\hat{\bar{v}}_1$, $\hat{\bar{v}}_2$, $\hat{\bar{p}}$, ${}_d({}_s\hat{\bar{\sigma}}_{xx})$, ${}_d({}_s\hat{\bar{\sigma}}_{yy})$, ${}_d({}_s\hat{\bar{\sigma}}_{xy})$, ${}_a\bar{\sigma}_{xy}$, $\hat{\bar{m}}_{zx}$, $\hat{\bar{m}}_{zy}$, and ${}^t\hat{\bar{\Theta}}_z$, hence this mathematical model has closure. The mathematical model is nondimensionalized using the following reference quantities. For length, density, velocity, viscosity, pressure and stress: $L_0, \rho_0, v_0, \eta_0, p_0, \tau_0$ in which $p_0 = \tau_0 = \rho_0 v_0^2$ (characteristic kinetic energy). This gives the following dimensionless quantities

$$\begin{gathered}
\bar{x} = \frac{\hat{\hat{x}}}{L_0}, \quad \bar{y} = \frac{\hat{\hat{y}}}{L_0}, \quad \bar{\rho} = \frac{\hat{\hat{\rho}}}{\rho_0}, \quad \bar{\eta} = \frac{\hat{\hat{\eta}}}{\eta_0}, \quad {}^m\bar{\eta} = \frac{{}^m\hat{\hat{\eta}}}{\eta_0} \\
{}_d({}_s\bar{\sigma}_{ij}) = \frac{{}_d({}_s\hat{\hat{\sigma}}_{ij})}{\tau_0}; \quad i = 1,2 \quad j = 1,2 \\
({}_a\bar{\sigma}_{yx}) = \frac{({}_a\hat{\hat{\sigma}}_{yx})}{\tau_0}, \quad \bar{p} = \frac{\hat{\hat{p}}}{p_0}, \quad m_0 = \tau_0 L_0
\end{gathered} \tag{4.32}$$

Using (4.32) in (4.24)–(4.31) we can obtain their following dimensionless form.

$$\bar{\rho}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}} + \frac{\partial \bar{v}_2}{\partial \bar{y}}\right) = 0 \tag{4.33}$$

$$\bar{\rho}\left(\bar{v}_1\frac{\partial \bar{v}_1}{\partial \bar{x}} + \bar{v}_2\frac{\partial \bar{v}_1}{\partial \bar{y}}\right) + \frac{\partial \bar{p}}{\partial \bar{x}} - \left(\frac{\partial {}_d({}_s\bar{\sigma}_{xx})}{\partial \bar{x}} + \frac{\partial {}_d({}_s\bar{\sigma}_{xy})}{\partial \bar{y}} - \frac{\partial({}_a\bar{\sigma}_{xy})}{\partial \bar{y}}\right) = 0 \tag{4.34}$$

$$\bar{\rho}\left(\bar{v}_1\frac{\partial \bar{v}_2}{\partial \bar{x}} + \bar{v}_2\frac{\partial \bar{v}_2}{\partial \bar{y}}\right) + \frac{\partial \bar{p}}{\partial \bar{y}} - \left(\frac{\partial {}_d({}_s\bar{\sigma}_{xy})}{\partial \bar{x}} + \frac{\partial {}_d({}_s\bar{\sigma}_{yy})}{\partial \bar{y}} - \frac{\partial({}_a\bar{\sigma}_{xy})}{\partial \bar{x}}\right) = 0 \tag{4.35}$$

$$\frac{\partial \bar{m}_{zx}}{\partial \bar{x}} + \frac{\partial \bar{m}_{zy}}{\partial \bar{y}} - 2({}_a\bar{\sigma}_{xy}) = 0 \tag{4.36}$$

$$\begin{aligned}
{}_d({}_s\bar{\sigma}_{xx}) &= \frac{1}{Re}2\eta\left(\frac{\partial \bar{v}_1}{\partial \bar{x}}\right) \\
{}_d({}_s\bar{\sigma}_{yy}) &= \frac{1}{Re}2\eta\left(\frac{\partial \bar{v}_2}{\partial \bar{y}}\right) \\
{}_d({}_s\bar{\sigma}_{xy}) &= \frac{1}{Re}\eta\left(\frac{\partial \bar{v}_1}{\partial \bar{y}} + \frac{\partial \bar{v}_2}{\partial \bar{x}}\right)
\end{aligned} \tag{4.37}$$

$$\bar{m}_{zx} = \frac{1}{ReL_0^2}{}^m\eta\left(\frac{\partial {}^t\bar{\Theta}_z}{\partial \bar{x}}\right) \tag{4.38}$$

$$\bar{m}_{zy} = \frac{1}{ReL_0^2}{}^m\eta\left(\frac{\partial {}^t\bar{\Theta}_z}{\partial \bar{y}}\right) \tag{4.39}$$

$${}^t\bar{\Theta}_z = \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{y}} + \frac{\partial \bar{v}_2}{\partial \bar{x}}\right) \tag{4.40}$$

Equations (4.33)–(4.40) are used for computing numerical solutions. Re is Reynolds number and is defined as $Re = \frac{\rho_0 L_0 v_0}{\eta_0}$.

A schematic of the cavity is shown in figure 4.10. The boundary conditions for dimensionless velocities $\bar{v}_1$ and $\bar{v}_2$ and pressure $\bar{p}$ are also shown in figure 4.10. Figure 4.10 shows a graded finite element discretization using 36 nine-node *p*-version finite elements [97]. The element sizes that the four corners are 0.05 units. The physical size of the cavity is 3 cm $\times$ 3 cm and the fluid properties are:

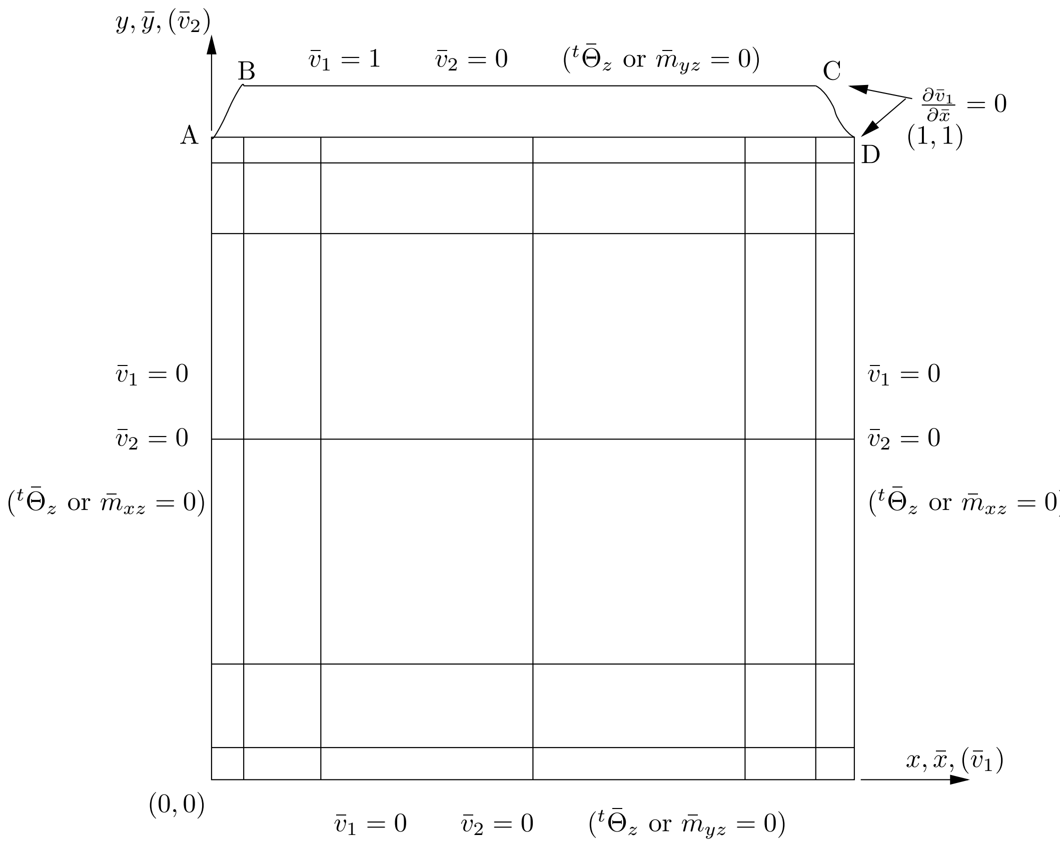


Figure 4.10: Schematic, discretization, boundary conditions, and lid velocity specification

$$\hat{\rho} = 998.2 \text{ kg/m}^3, \qquad \hat{\eta} = 1.002 \times 10^{-3} \text{ Pa}$$

We consider the following reference values:

$$\rho_0 = \hat{\rho} = 998.2 \text{ kg/m}^3, \quad \eta_0 = \hat{\eta} = 1.002 \times 10^{-3} \text{ Pa}$$
$$L_0 = 0.003 \text{ m}, \quad v_0 = 0.03346 \text{ m/s}$$

With this choice of reference values, we have

$$\rho = 1, \quad \eta = 1, \quad Re = \frac{\rho_0 \, L_0 \, v_0}{\eta_0} = 1000$$

and the dimensionless cavity is 1×1.

The velocity of the lid is assumed to vary from zero at the vertical walls to one in a continuous and differentiable manner over a length of h_d representing the characteristic lengths of the elements at the top two corners. Using the conditions

$$\bar{v}_1 = 0, \quad \frac{\partial \bar{v}_1}{\partial \bar{x}} = 0 \quad \text{at} \quad \bar{x} = \bar{x}_A = 0 \text{ and } \bar{x} = \bar{x}_B = h_d$$

we can obtain a cubic distribution of $\bar{v}_1$ over $0 \leq x \leq h_d$ that is continuous and differentiable, and likewise using

$$\bar{v}_1 = 0, \quad \frac{\partial \bar{v}_1}{\partial \bar{x}} = 0 \quad \text{at} \quad \bar{x} = \bar{x}_C = 1 - h_d \text{ and at } \bar{x} = \bar{x}_D = 1$$

we can obtain another cubic distribution of $\bar{v}_1$ over $1 - h_d \leq x \leq 1$ that is continuous and differentiable.

The smaller element sizes at the top two corners are intentionally chosen so that a good approximation of constant lid velocity can be obtained by the v_1 velocity distributions shown in figure 4.10. We consider equal degree, equal order local approximations of all dependent variables over each element of the discretization. Solutions of class $C^{11}(\bar{\Omega}^e)$ at uniform p-levels of 7 in ξ and η directions are computed for all values of η and ${}^m\eta$. Refer to [97–104] for details on Newton's linear method, its convergence, and the residual functional based on integrated sum of squares of the residuals (I) from each equation. At p-level of 7, I values of $O(10^{-8})$ are obtained except the two elements in the top corners (as expected). In all computations $\eta = 1$ (dimensionless viscosity) is used and ${}^m\eta$ is varied from $0.0001 - 0.01$ (fractions of η). Additional boundary conditions are needed for the polar cases, and we must choose to constrain the moment $\bar{M}_z$ or rate of rotation ${}^t\bar{\Theta}_z$. In this study we consider both cases, first ${}^t\bar{\Theta}_z = 0$ which represents the case where the fluid particles at the wall are completely restricted from rotating, and also where $\bar{M}_z = 0$ represents the case where the fluid particles are free to rotate against the boundary. This corresponds physically to a boundary which is able exert arbitrarily large moments on the fluid in the constrained rotation case, or is unable to exert any moments on the fluid in the constrained moment case. The choice of boundary conditions in an application of this theory should of course be guided by the physics of the boundary of interest.

Since the mathematical model is a system of first order partial differential equations, the choice of $C^{11}(\bar{\Omega}^e)$ local approximations ensure that all integrals over the discretization are Riemann. With this feature, when $I \to 0$ we are ensured that the PDEs are satisfied in the pointwise sense.

Figures 4.12 and 4.14 show plots of velocity $\bar{v}_1$ versus $\bar{y}$ at $\bar{x} = 0.5$ and velocity $\bar{v}_2$ versus $\bar{x}$ at $\bar{y} = 0.5$ for $\eta = 1$ and ${}^m\eta = 0.0001 - 0.05$. Non-polar results ($\eta = 1, {}^m\eta = 0$) are in good agreement with published results [105]. When ${}^m\eta$ is non-zero the velocity profiles in figures 4.12 and 4.14 change significantly. Progressively increasing values of ${}^m\eta$ offer progressively increasing resistance to the propagation of disturbance from the lid into the cavity, as a consequence in the immediate vicinity of the lid, the velocity $\bar{v}_1$ drops smoothly (figure 4.12), peak negative values of $\bar{v}_1$ shifts upwards compared to the non-polar case and peak values of $\bar{v}_2$ drop and shift to the right of the left wall and left of the right wall (figure 4.14). The consequence of this is that velocities drop from the lid more smoothly for progressively increasing values of ${}^m\eta$ and the center of the circulation moves towards the lid with reduction in its size. We can observe this behavior in the contour and carpet plots of total velocity field (scalar) in figures 4.27–4.33. The consequence of the choice of moment-free or constrained rotation boundary conditions are most apparent at small values of ${}^m\eta$. When the boundary is moment-free, the polar solution approaches the non-polar solution, however when the rotations are constrained the velocity gradients near the boundary in the polar case never approach those in the non-polar case. Figures 4.16–4.20 show plots of ${}_d({}_s\bar{\sigma}_{xx})$, ${}_d({}_s\bar{\sigma}_{yy})$, and ${}_d({}_s\bar{\sigma}_{xy})$ versus distance $\bar{y}$ at $\bar{x} = 0.5$. Minor oscillations in the solutions are due to coarse mesh and low p-level, however the oscillations are less severe when the moment-free boundary condition is used. The solutions in figures 4.16 and 4.18 satisfy ${}_d({}_s\bar{\sigma}_{xx}) +_d ({}_s\bar{\sigma}_{yy}) = 0$ quite well (as required by continuity) numerically. Values of ${}_d({}_s\bar{\sigma}_{xx})$ and ${}_d({}_s\bar{\sigma}_{yy})$ progressively reduce with increasing ${}^m\eta$ while ${}_d({}_s\bar{\sigma}_{xy})$ increases. Antisymmetric stress ${}_a(\bar{\sigma}_{yx})$ (figure 4.22) versus $\bar{y}$ at $\bar{x} = 0.5$ shows progressively increasing values with increasing ${}^m\eta$ as expected due to increasing resistance to the flow. Figures 4.24 and 4.26 show plots of moments $\bar{m}_{zx}$ and $\bar{m}_{zy}$ versus $\bar{y}$ at $\bar{x} = 0.5$. As expected, increasing values of ${}^m\eta$ results in progressively increasing values of the moments.

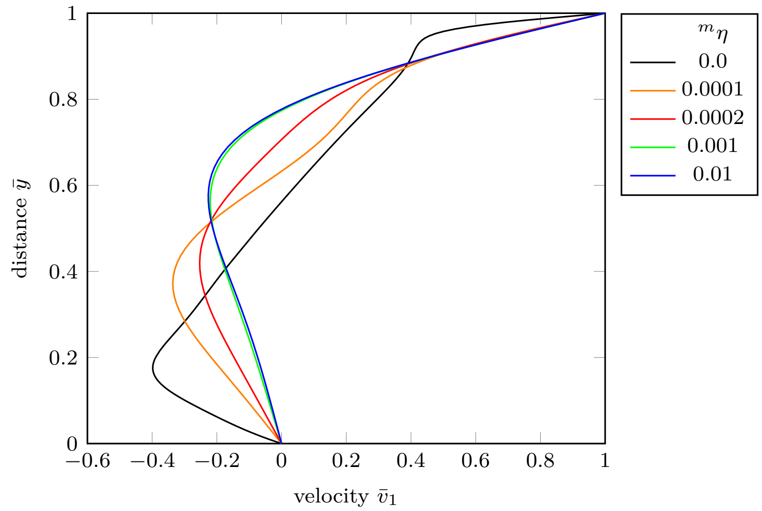


Figure 4.11: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

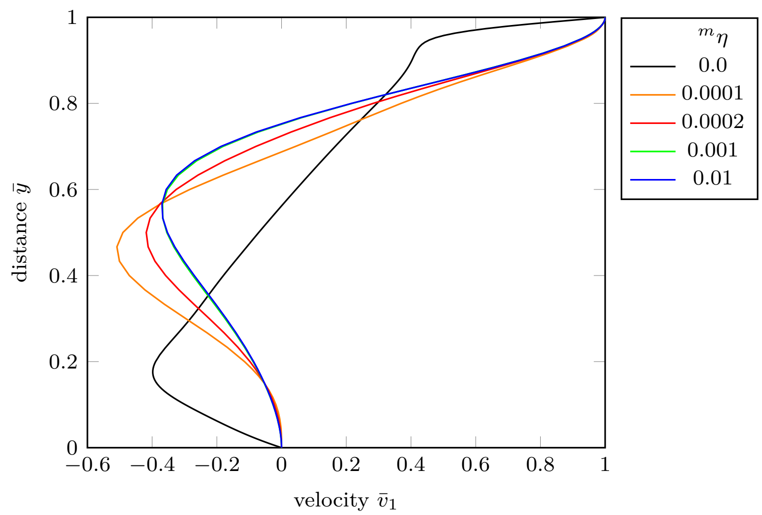


Figure 4.12: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 0.5$

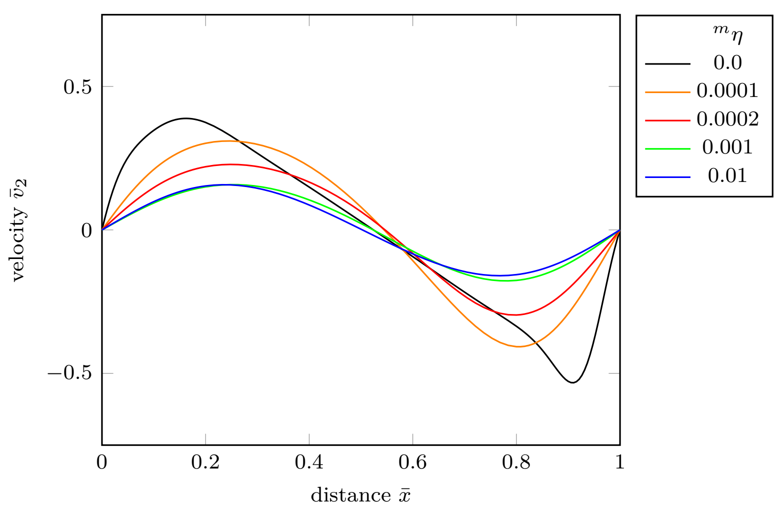


Figure 4.13: Velocity $\bar{v}_2$ versus distance $\bar{x}$ at $\bar{y} = 0.5$: moment-free boundary

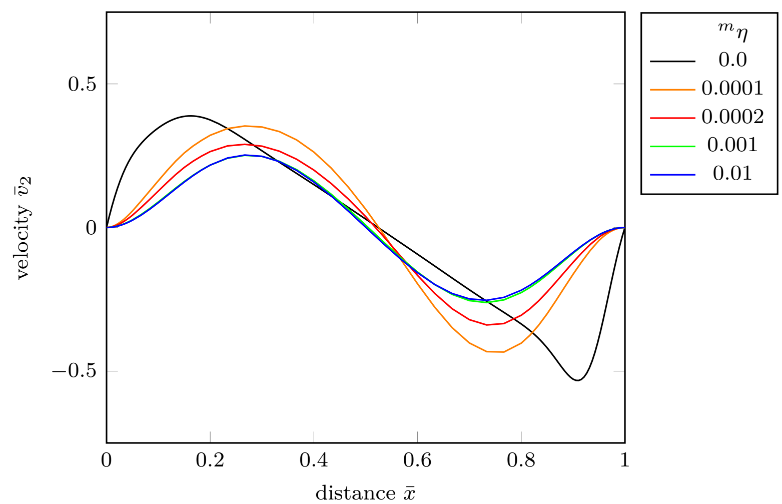


Figure 4.14: Velocity $\bar{v}_2$ versus distance $\bar{x}$ at $\bar{y} = 0.5$

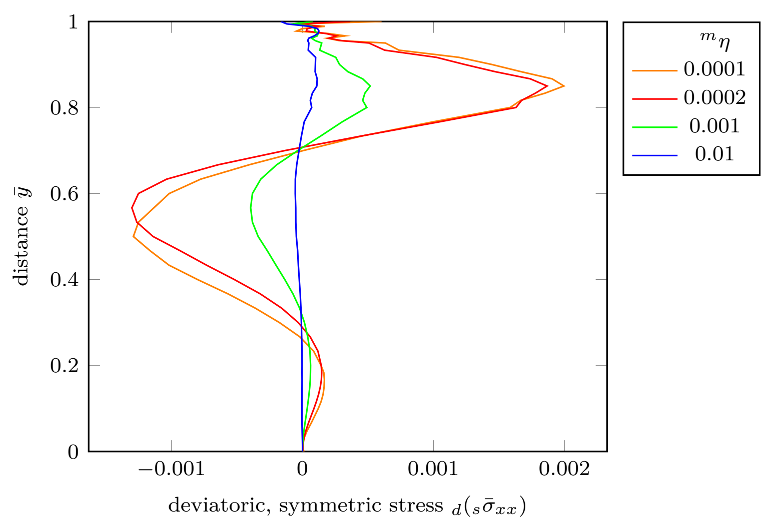


Figure 4.15: Stress ${}_{d}({}_{s}\bar{\sigma}_{xx})$ versus $\bar{y}$ at $\bar{x} = 0.5$

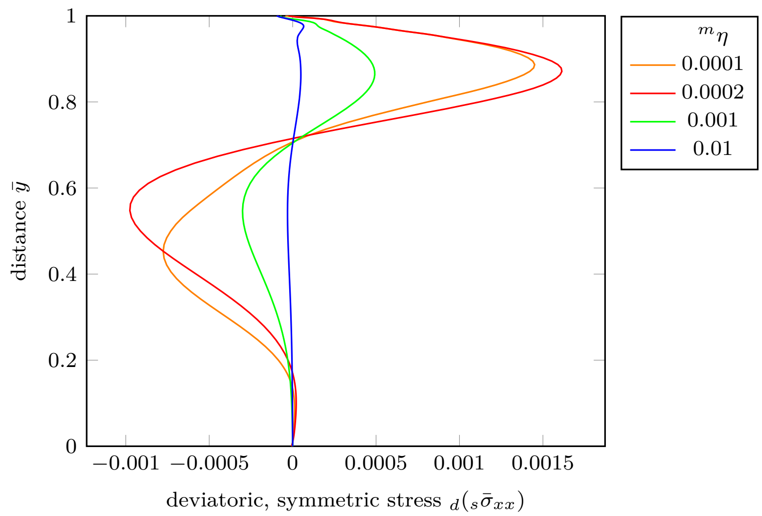


Figure 4.16: Stress ${}_d({}_s\bar{\sigma}_{xx})$ versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

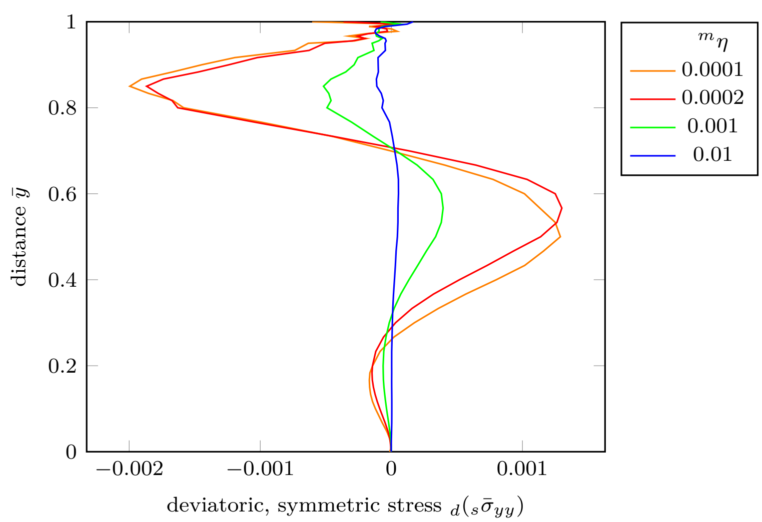


Figure 4.17: Stress ${}_d({}_s\bar{\sigma}_{yy})$ versus $\bar{y}$ at $\bar{x} = 0.5$

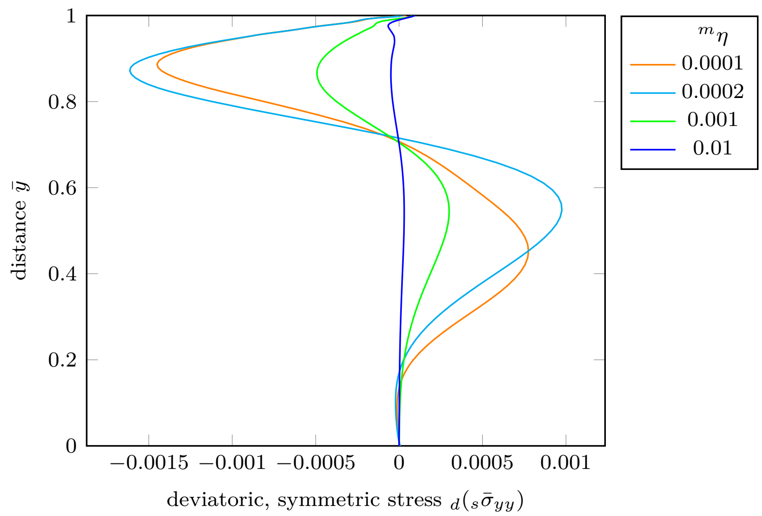


Figure 4.18: Stress $_d(_s\bar{\sigma}_{yy})$ versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

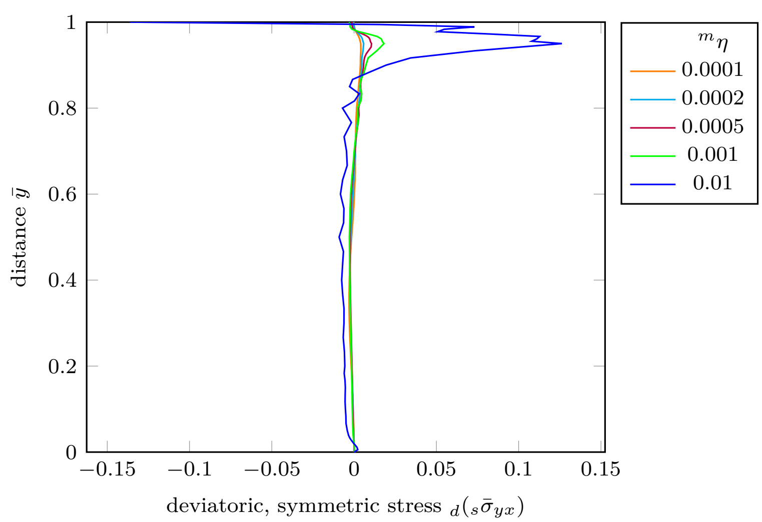


Figure 4.19: Stress $_d(_s\bar{\sigma}_{yx})$ versus $\bar{y}$ at $\bar{x} = 0.5$

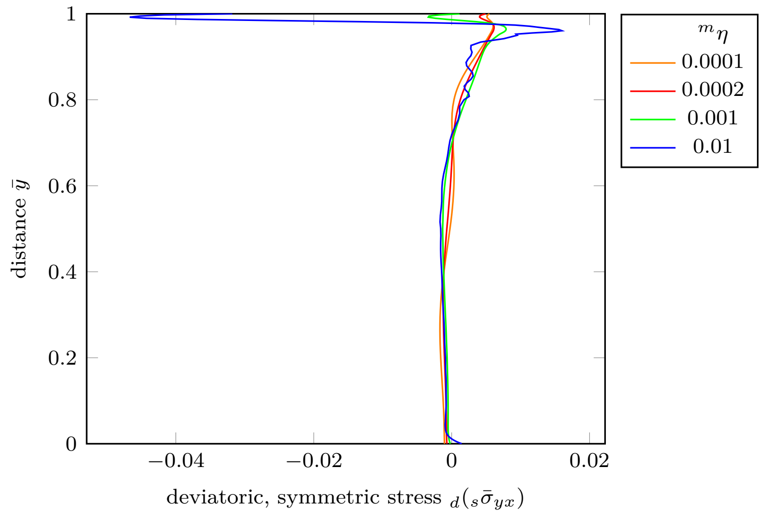


Figure 4.20: Stress ${}_d({}_s\bar{\sigma}_{yx})$ versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

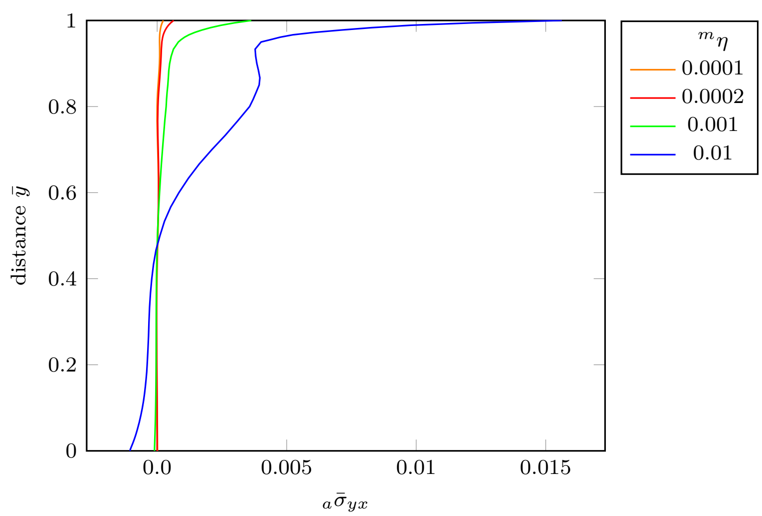


Figure 4.21: Stress $({}_a\bar{\sigma}_{yx})$ versus $\bar{y}$ at $\bar{x} = 0.5$

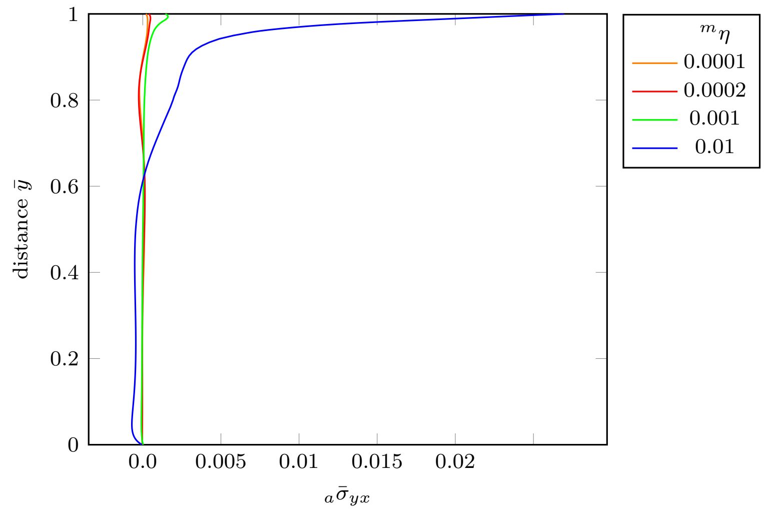


Figure 4.22: Stress (${}_a\bar{\sigma}_{yx}$) versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

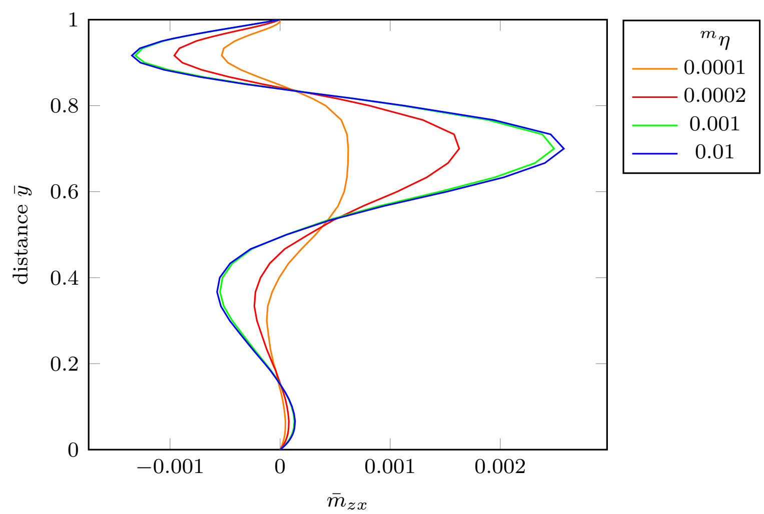


Figure 4.23: Moment $\bar{m}_{zx}$ versus $\bar{y}$ at $\bar{x} = 0.5$

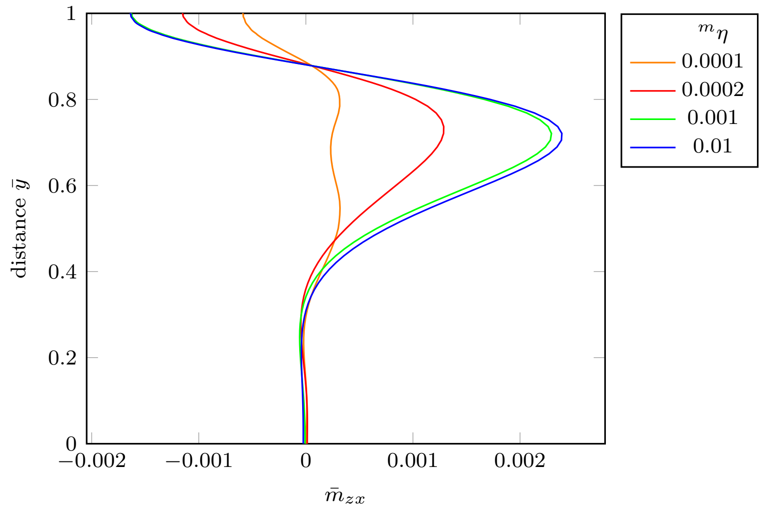


Figure 4.24: Moment $\bar{m}_{zx}$ versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

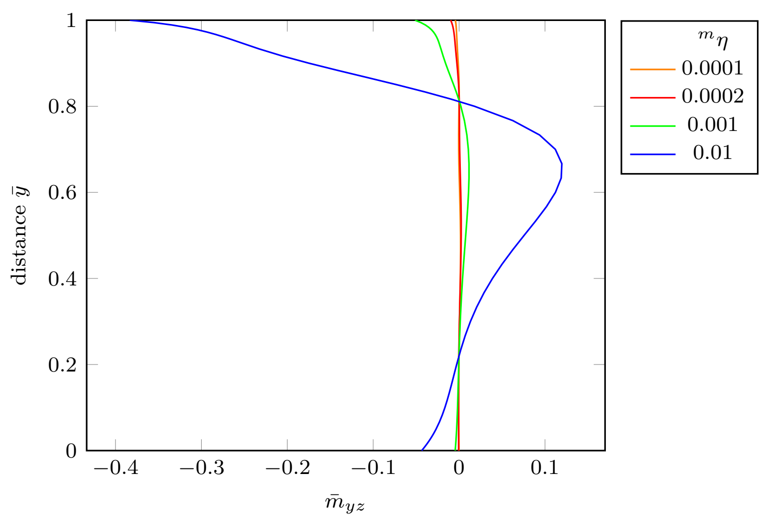


Figure 4.25: Moment $\bar{m}_{zy}$ versus $\bar{y}$ at $\bar{x} = 0.5$

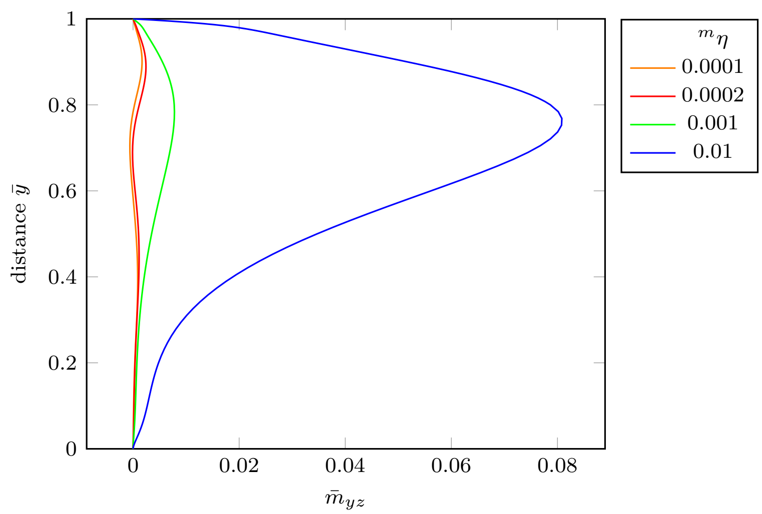


Figure 4.26: Moment $\bar{m}_{zy}$ versus $\bar{y}$ at $\bar{x} = 0.5$: moment-free boundary

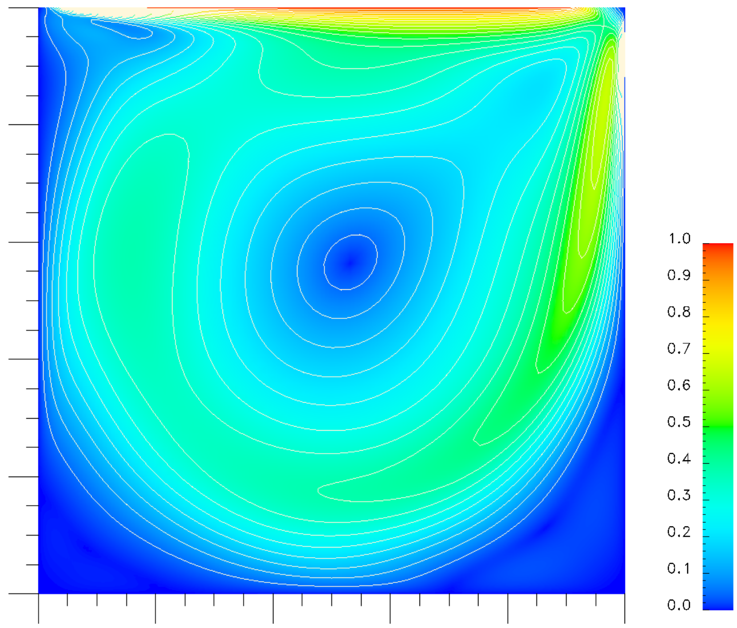


Figure 4.27: Velocity contour plots: ${}^{m}\eta = 0$

4.1.3 Model problem 3: Flow past a sudden expansion

In this model problem we consider isothermal flow of an incompressible thermoviscous polar past a sudden expansion or backward-facing step. As in the previous section, we will compare the results from the non-polar case to the polar case for varying polar viscosity ${}^{m}\eta$. The solution for the non-polar case has been studied extensively in the literature, including both numeric [102] and experimental results. The problem is of particular interest here due to the high gradients of rotation rates near the recirculation zone, which will result in additional resistance to flow when the polar physics is included.

The mathematical model is the same as in section 4.1.2, equations (4.33)–(4.40). Figure 4.34 shows the domain and boundary conditions for the problem. The inlet boundary conditions consist of an applied velocity profile corresponding to fully developed flow between parallel plates. For the non-polar case the theoretical parabolic solution is used, while the polar cases use velocity profiles computed from model problem 1 in section 4.1.1. The inlet volumetric flow rate is kept at a constant value of 1.0 (dimensionless) for all studies, resulting in flow at a constant Reynolds number of $Re = 229$. Outlet boundary conditions are fully developed: $v = 0$, $\bar{\sigma}_{xx} = 0$, $m_{xz} = 0$, and derivatives of all dependent variables with respect to x are set to 0. The top and bottom boundary conditions are the same as in the bottom boundary of model problem 2: $\bar{v}_1 = \bar{v}_2 = 0$ and either ${}^{t}\bar{\Theta}_z = 0$ or $\bar{m}_{yz} = 0$. The vertical boundary at the expansion uses the same boundary conditions as the left and right boundaries in model problem 2: $\bar{v}_1 = \bar{v}_2 = 0$ and either ${}^{t}\bar{\Theta}_z = 0$ or $\bar{m}_{xz} = 0$. The same physical constants as model problem 2 are used, and again the polar viscosity (${}^{m}\eta$) is varied

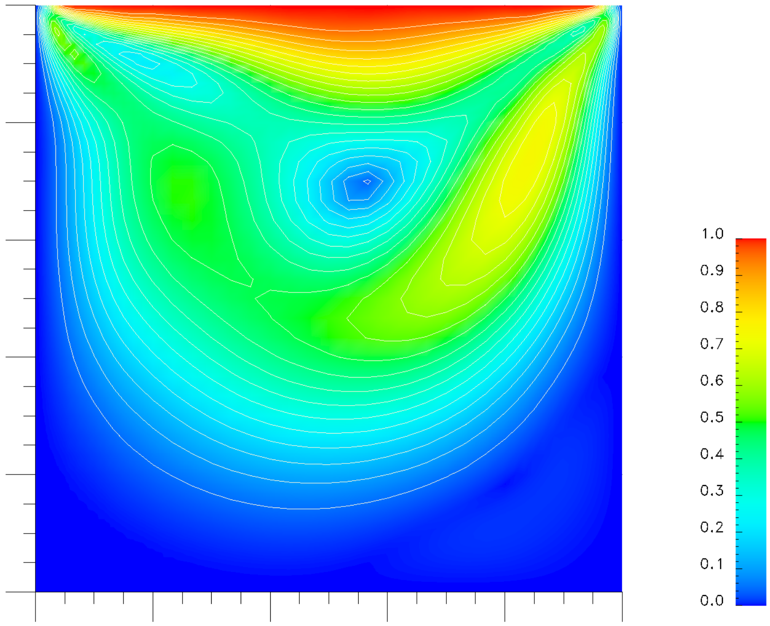


Figure 4.28: Velocity contour plots: ${}^{m}\eta = 0.0001$

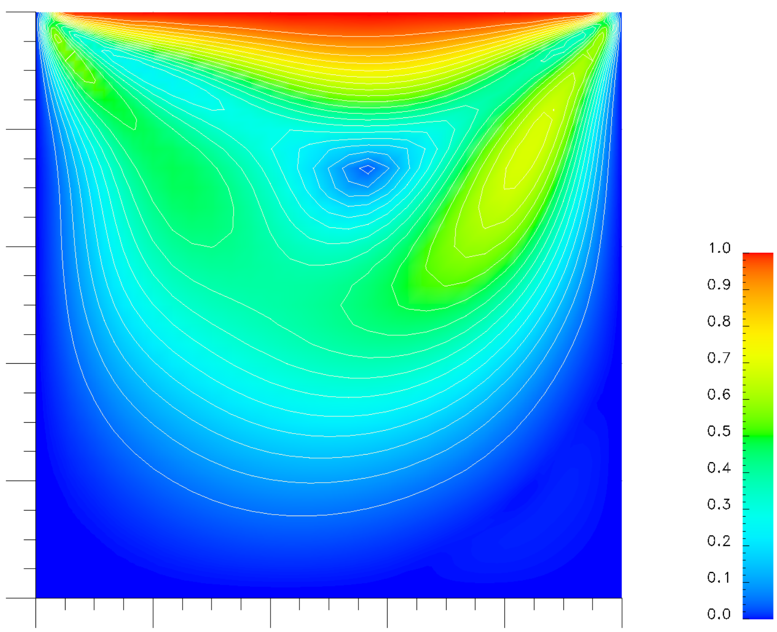


Figure 4.29: Velocity contour plots: ${}^{m}\eta = 0.0002$

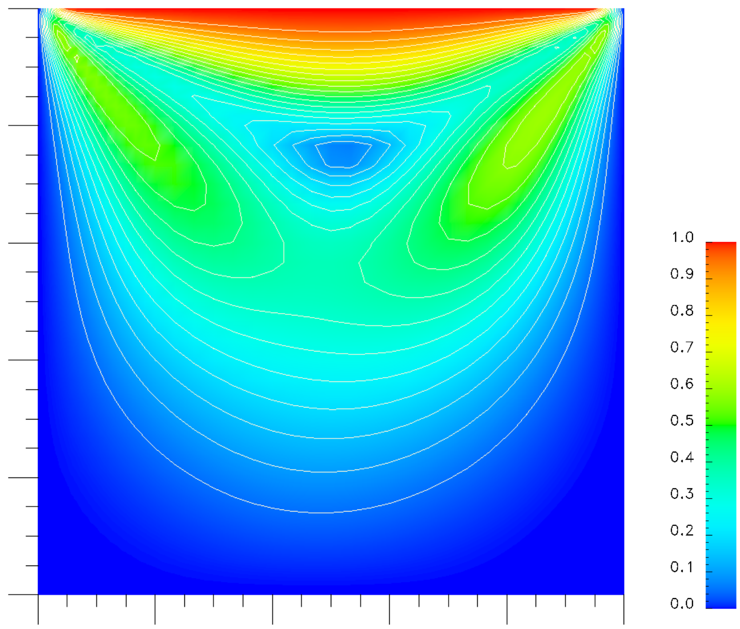


Figure 4.30: Velocity contour plots: ${}^{m}\eta = 0.001$

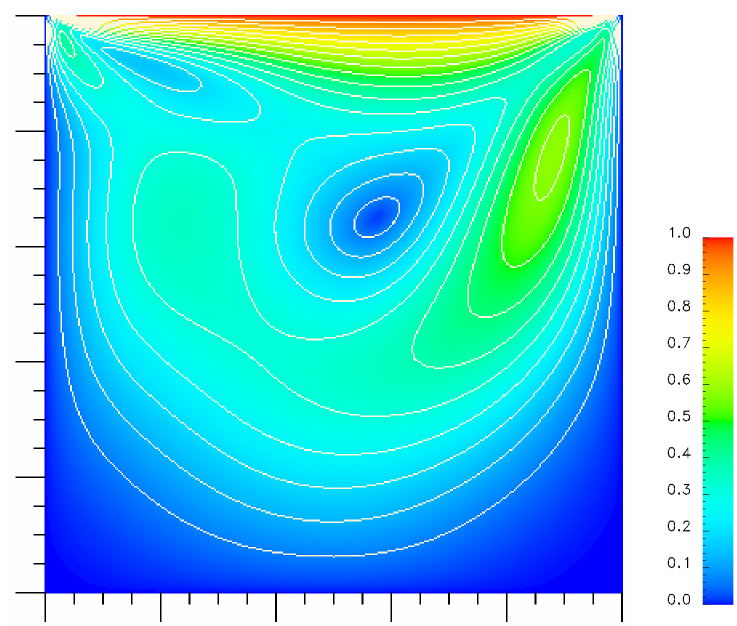


Figure 4.31: Velocity contour plots: ${}^{m}\eta = 0.0001$: moment-free boundary

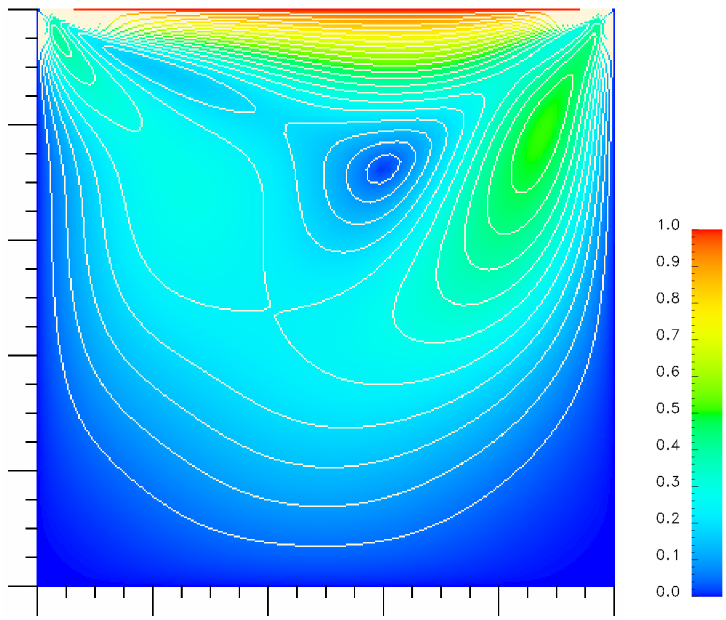


Figure 4.32: Velocity contour plots: ${}^{m}\eta = 0.0002$: moment-free boundary

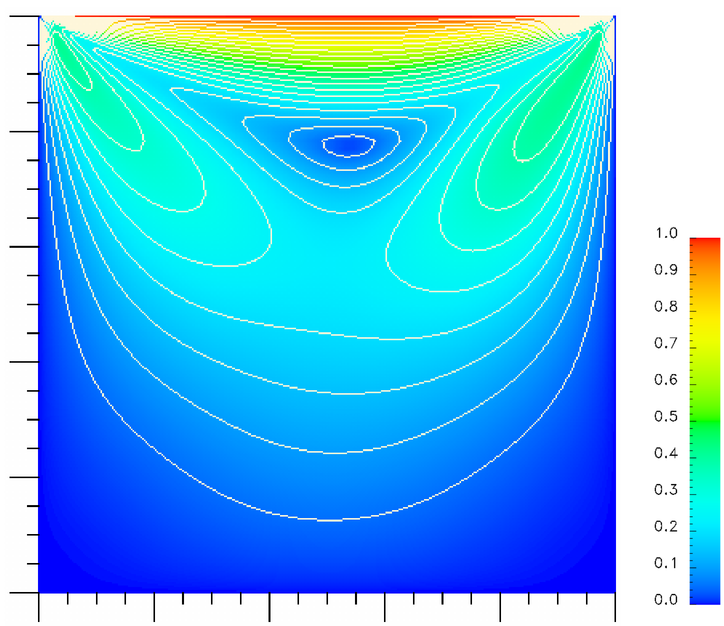


Figure 4.33: Velocity contour plots: ${}^{m}\eta = 0.001$: moment-free boundary

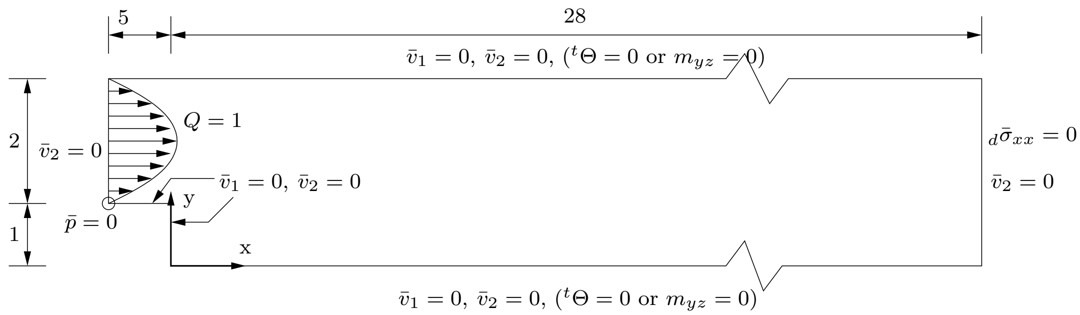


Figure 4.34: Schematic, boundary conditions, and inlet velocity specification

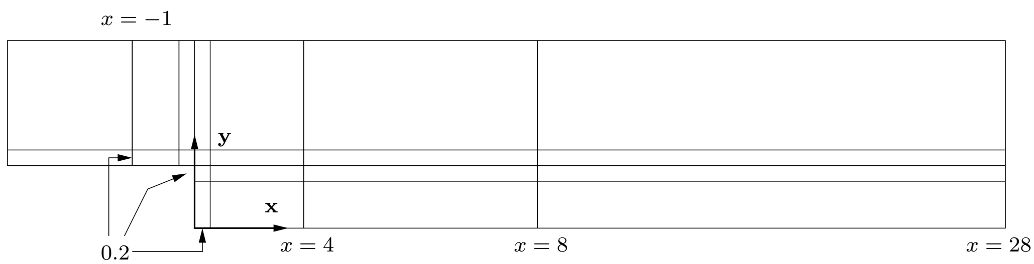


Figure 4.35: Discretization details for model problem 3

as a fraction of the Newtonian viscosity (η).

Numeric solutions are computed using the least squares finite element process. The domain is discretized using 42 elements as shown in figure 4.35. Approximations of class $C^{11}(\Omega)$ are used with a p-level of 9 for all variables. This results in values of the least squares functional $I = O(10^{-6})$ or better for most elements, with the exception of the elements near the interior corner. The least squares functional is 10^{-2} for these elements, and is an unavoidable consequence of the geometry of the domain. This problem is also observed in solutions for non-polar fluids, and even the theoretical solution for Stokes flow has a singularity at that point. Figure 4.36 shows the values of the least-squares functional for each element in the case where ${}^m\eta = 0.01\eta$, all other values of ${}^m\eta$ resulted in better residuals. We note that while we are able to obtain similar magnitudes of residuals for the polar case as the non-polar case with the same p-level, obtaining converged results for the polar case required increasing the inlet length compared to the mesh that was used in [102].

The results for this model problem at the inlet $x = -5.0$ (figures 4.37 and 4.40) show little difference in the flow field as ${}^m\eta$ increases. For moment free boundary condition case the flow field is almost identical to the non-polar results, while in the constrained rotation boundary condition case the flow field is obviously different due to the different kinematic boundary condition. Note that these results differ from what was observed in Model Problem 1 due to the fact that this study is at constant flow rate, while Model Problem 1 was at constant pressure gradient. The differences

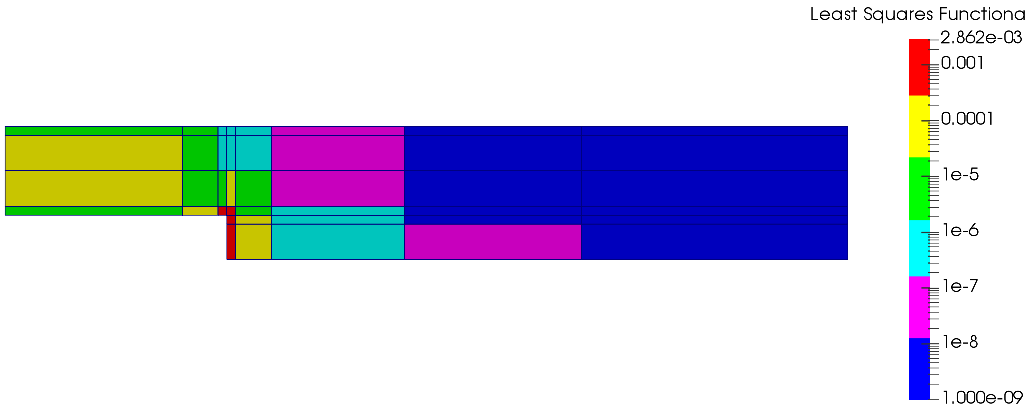


Figure 4.36: Element-wise residual functionals for ${}^{m}\eta = 0.01\eta$

in the flow field begin to appear at the expansion $x = 0.0$, as shown in figures 4.38 and 4.41, and the greatest difference is observed just past the expansion at $x = 1.0$. The results at $x = 1.0$ (figures 4.39 and 4.42) show that as ${}^{m}\eta$ increases the amount of recirculation decreases. For low values of ${}^{m}\eta$ the magnitude of the negative velocity is less than the non-polar case, and for high values of ${}^{m}\eta$ the negative velocity region is no longer observed at this location. Figures 4.43–4.47 show fringe plots of the x-velocity component $\bar{v}_1$. We observe that the length of the recirculation zone decreases as ${}^{m}\eta$ increases due to the additional resistance to changing rates of rotation.

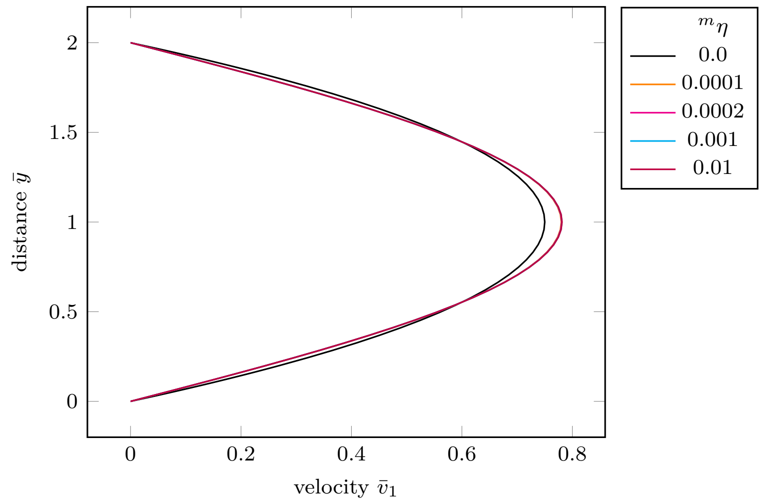


Figure 4.37: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at the inlet: moment-free boundary

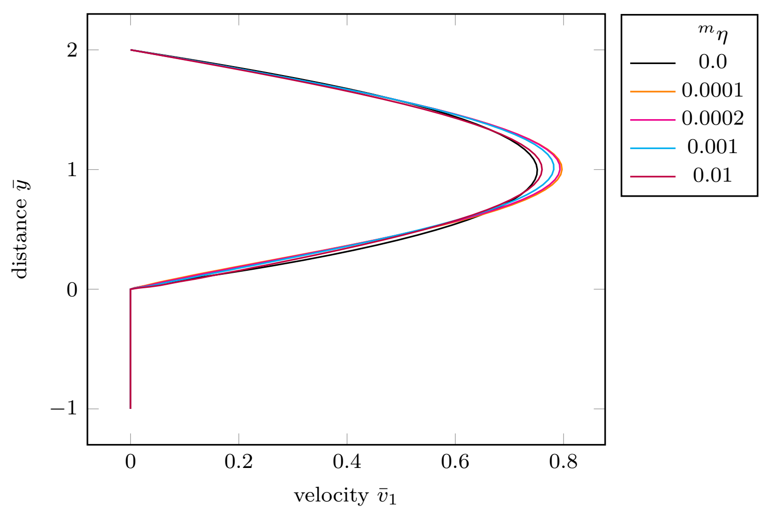


Figure 4.38: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 0.0$: moment-free boundary

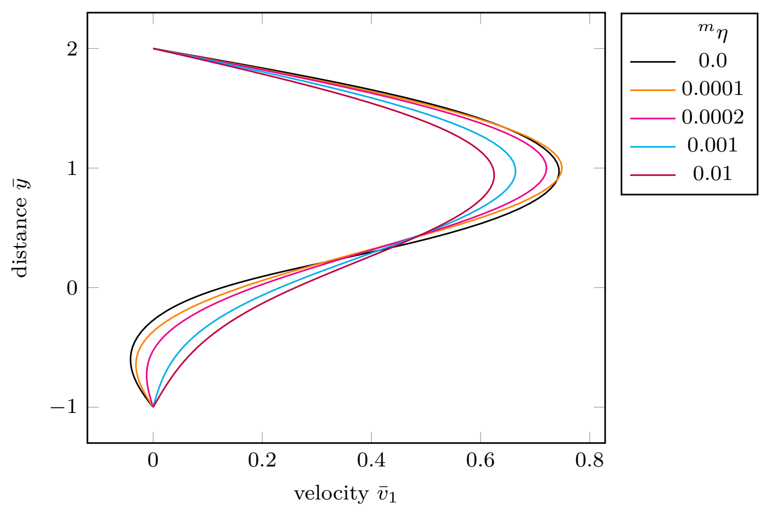


Figure 4.39: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 1.0$: moment-free boundary

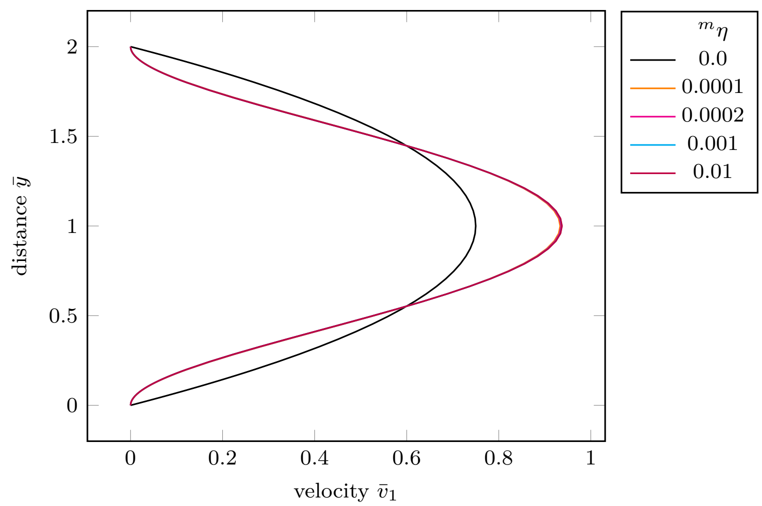


Figure 4.40: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at the inlet

Remarks

From the three model problems we clearly observe the consequences of internal polar physics when compared with the non-polar case for thermofluids.

1. Increasing values of ${}^m\eta$ offer progressively increasing resistance to fluid motion

2. For a given $\frac{\partial \bar{p}}{\partial \bar{y}}$, the flow can be completely choked by increasing ${}^m\eta$ while holding η constant in the case of flow between parallel plates. This clearly demonstrates the additional resistance to flow offered by the internal polar physics.

3. Behavior similar to flow between parallel plates is also observed for lid driven square cavity. With progressively increasing resistance to fluid motion for progressively increasing ${}^m\eta$, hence progressively more pronounced influence of internal polar physics: (a) the lid velocity propagates more smoothly into the cavity (b) the circulation zone moves progressively towards the lid (c) the circulation zone size reduces compared to non-polar behavior (${}^m\eta = 0$, $\eta = 1$).

4. The sudden expansion also shows increasing resistance to fluid motion with increasing ${}^m\eta$. The length of the recirculation zone decreases as the effect of the polar physics increases.

5. Pronounced influence of the internal polar physics on the behavior of the flow is clearly observed in all three model problems.

4.2 Model problems and solutions for internal polar solid continua

In this section we consider simple model problems in which the non-polar physics is well understood so that the influence of internal polar physics on the deformation behavior can be clearly demonstrated. We consider conservation and balance laws in $\mathbb{R}^2$ for plane stress behavior. This

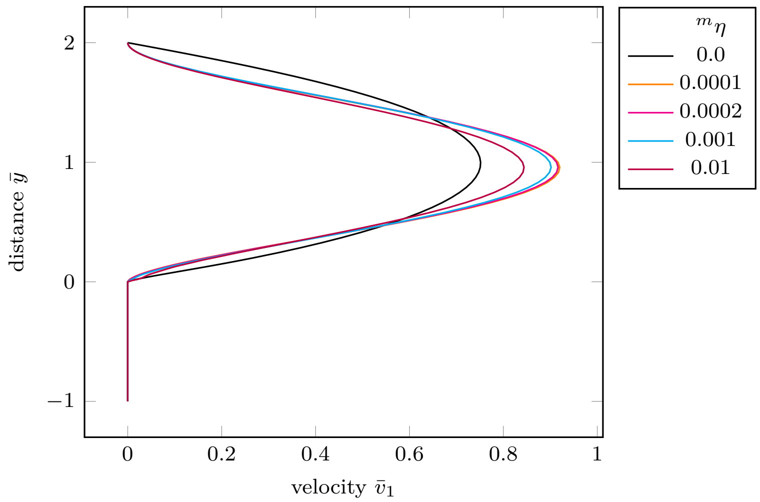


Figure 4.41: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 0.0$

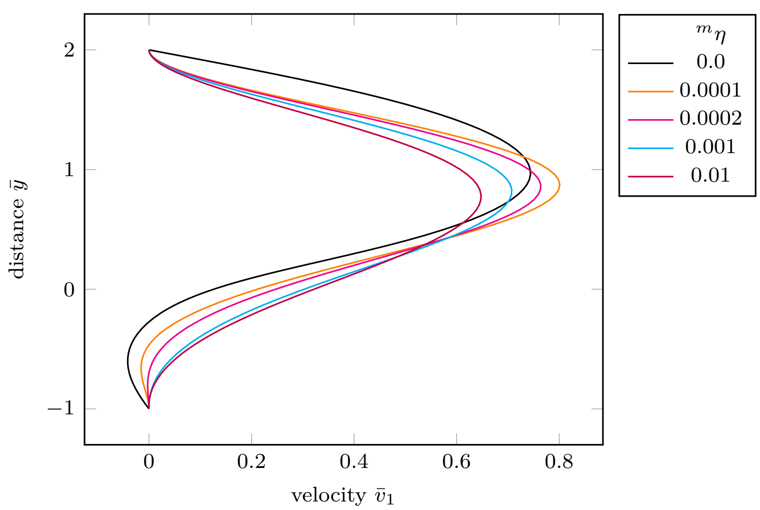


Figure 4.42: Velocity $\bar{v}_1$ versus distance $\bar{y}$ at $\bar{x} = 1.0$

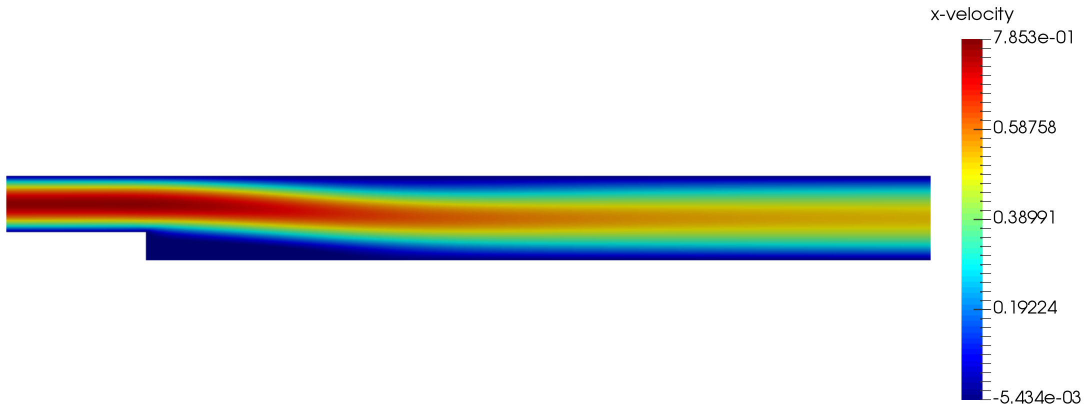


Figure 4.43: x-velocity plot: ${}^{m}\eta = 0$

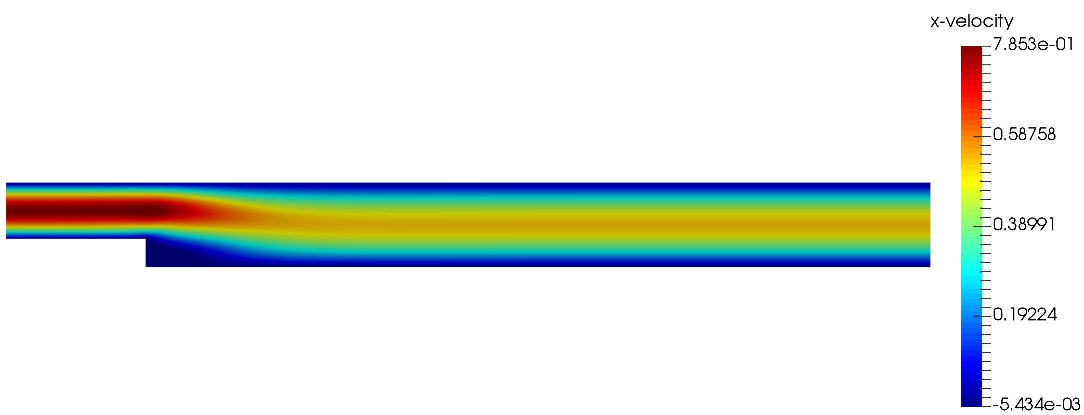


Figure 4.44: x-velocity plot: ${}^{m}\eta = 0.0001$: moment-free boundary

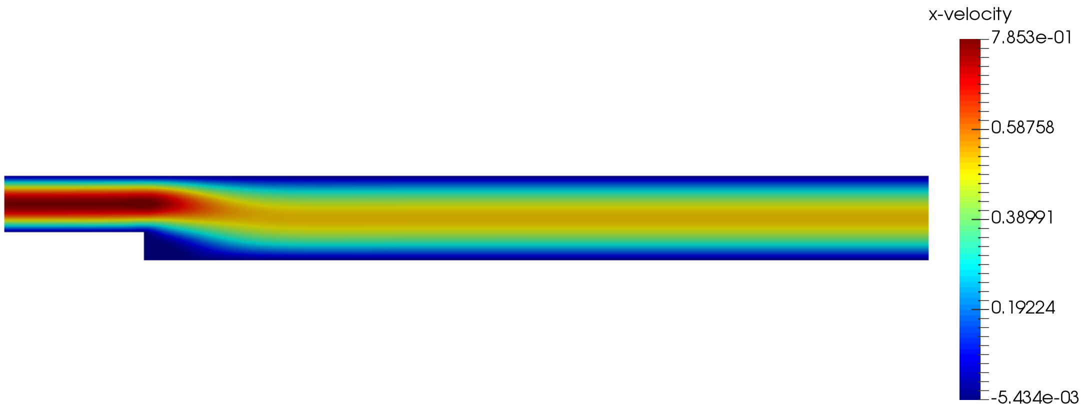


Figure 4.45: x-velocity plot: ${}^{m}\eta = 0.0002$: moment-free boundary

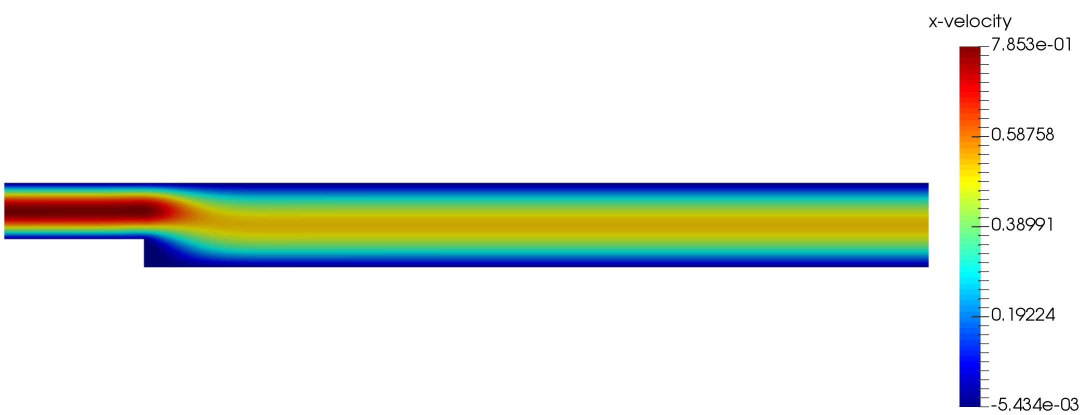


Figure 4.46: x-velocity plot: ${}^{m}\eta = 0.001$: moment-free boundary

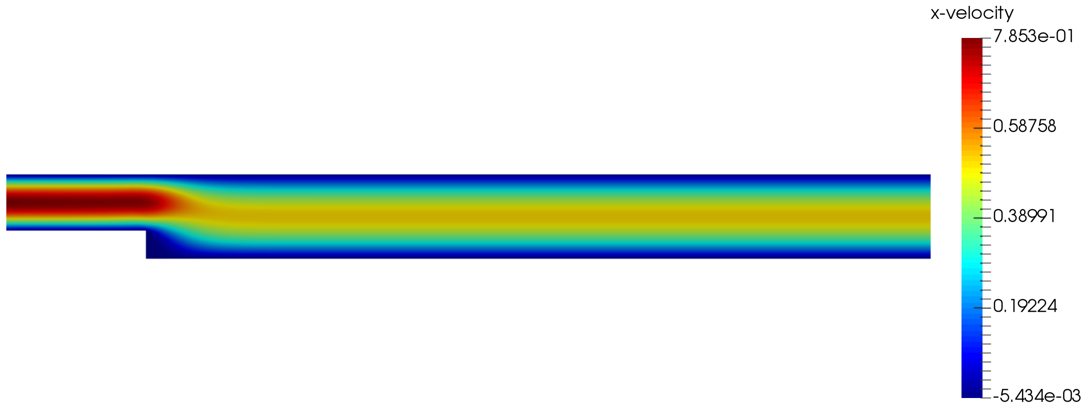


Figure 4.47: x-velocity plot: ${}^{m}\eta = 0.01$: moment-free boundary

mathematical model is used to study non-internal polar i.e. classical and internal polar physics and its influence on deformation of a clamped-clamped plate and a simply supported plate.

4.2.1 Mathematical model in $\mathbb{R}^3$

Following references [96, 106] the conservation and balance laws (conservation of mass, balance of linear momenta, balance of angular momenta, balance of moments of moments, first and second laws of thermodynamics) in $\mathbb{R}^3$ for internal polar thermoelastic solid continua with small deformation and small strain in Lagrangian description can be written (using modified Helmholtz free energy density $\underset{\sim}{\Phi}$ and modified specific internal energy $\underset{\sim}{e}$) as

$$
\begin{gathered}
\rho_0 = |J|\rho(\boldsymbol{x},t) \\
\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma}) = 0 \\
m_{mk,m} - \epsilon_{ijk}(\sigma_{ij}) = 0 \\
\epsilon_{ijk} m_{ij} = 0 \\
\rho_0 \frac{D\underset{\sim}{e}}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \\
\rho_0 \left(\frac{D\underset{\sim}{\Phi}}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} \leq 0 \\
\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}
\end{gathered}
\tag{4.41}
$$

If we consider the stress decomposition

$$\boldsymbol{\sigma} =_s \boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \tag{4.42}$$

in which ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$ are symmetric and antisymmetric stress tensors, then using

$$\epsilon_{ijk}\sigma_{ij} = \epsilon_{ijk}\left({}_a\sigma_{ij}\right) \tag{4.43}$$

and (4.42) in (4.41), the conservation and balance laws can be written as (Substituting for $\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}$)

$$\begin{aligned}
\rho_0 &= |J|\rho \\
\rho_0 \frac{D^2\boldsymbol{u}}{Dt^2} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) &= 0 \\
m_{mk,m} - \epsilon_{ijk}({}_a\sigma_{ij}) &= 0 \\
\epsilon_{ijk} m_{ij} &= 0 \\
\rho_0 \frac{D\underset{\sim}{e}}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} &= 0 \\
\rho_0\left(\frac{D\underset{\sim}{\Phi}}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} &\leq 0
\end{aligned} \tag{4.44}$$

Using the constitutive theories derived in this paper, we consider the following for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$ and $\boldsymbol{q}$ (equations (3.256), (3.259), and (3.271) in the absence of $(\theta - \theta_{\underline{\Omega}})$ term)

$$[{}_s\sigma] = 2\mu[\varepsilon] + \lambda \mathrm{tr}[\varepsilon] \tag{4.45}$$

$$[m] = 2\mu_m[{}^{\Theta}_{s}J] + \lambda_m \mathrm{tr}[{}^{\Theta}_{s}J] \tag{4.46}$$

$$\{q\} = -k\,\{g\} \tag{4.47}$$

In which

$$[{}^{\Theta}_{s}J] = \frac{1}{2}\left([{}^{\Theta}J] + [{}^{\Theta}J]^T\right) \tag{4.48}$$

$$[{}^{\Theta}J] = \frac{\partial\{\Theta\}}{\partial\{x\}} \quad \text{or} \quad {}^{\Theta}J_{ij} = \frac{\partial\Theta_i}{\partial x_j} \tag{4.49}$$

$$\{\Theta\}^T = [\Theta_{x_1}, \Theta_{x_2}, \Theta_{x_3}] \tag{4.50}$$

$$\begin{aligned}
\Theta_{x_1} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\
\Theta_{x_2} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \\
\Theta_{x_3} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)
\end{aligned} \tag{4.51}$$

$$[\varepsilon] = \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) \tag{4.52}$$

$$[{}^dJ] = \frac{\partial\{u\}}{\partial\{x\}} \quad \text{or} \quad {}^dJ_{ij} = \frac{\partial u_i}{\partial x_j} \tag{4.53}$$

$$g_i = \frac{\partial\theta}{\partial x_i} \tag{4.54}$$

μ, λ, μ_m, λ_m and k are material coefficients.

4.2.2 Mathematical model in $\mathbb{R}^2$

For the sake of convenience we choose x_1, x_2 as x, y; and express the material derivative of $\boldsymbol{v}$ in the linear momentum equations in terms of displacements (i.e. use balance of linear momenta in (4.44)). For small deformation, $|J| \approx 1$, hence $\rho_0 = \rho$ i.e. the solid continua is not compressible. The mathematical model in section 4.2.1 in $\mathbb{R}^3$ can be reduced to (using $\frac{D}{Dt} = \frac{\partial}{\partial t}$ in Lagrangian description) $\mathbb{R}^2$, keeping in mind that ${}_a\sigma_{xy} = -{}_a\sigma_{yx}$ and for boundary value problems the inertial terms in the linear momentum equations are absent. We further assume the body forces to be absent.

Conservation and balance laws:

$$\begin{aligned}
\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} + \frac{\partial({}_a\sigma_{yx})}{\partial y} &= 0 \\
\frac{\partial({}_s\sigma_{xy})}{\partial y} + \frac{\partial({}_s\sigma_{yy})}{\partial y} - \frac{\partial({}_a\sigma_{yx})}{\partial x} &= 0 \\
\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) &= 0 \\
\rho_0\frac{\partial \underset{\sim}{e}}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} &= 0 \\
\rho_0\left(\frac{\partial \underset{\sim}{\Phi}}{\partial t} + \eta\frac{\partial\theta}{\partial t}\right) + q_x\frac{\partial\theta}{\partial x} + q_y\frac{\partial\theta}{\partial y} &\leq 0
\end{aligned} \tag{4.55}$$

Constitutive theories:

$$\begin{aligned}
{}_s\sigma_{xx} &= D_{11}\frac{\partial u_1}{\partial x} + D_{12}\frac{\partial u_2}{\partial y} \\
{}_s\sigma_{yy} &= D_{21}\frac{\partial u_1}{\partial x} + D_{22}\frac{\partial u_2}{\partial y}; \quad D_{21} = D_{12} \\
{}_s\sigma_{xy} &= D_{33}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\
m_{xz} &= E_m\frac{\partial}{\partial x}\left(\Theta_z\right) \\
m_{yz} &= E_m\frac{\partial}{\partial y}\left(\Theta_z\right) \\
\Theta_z &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x}\right)
\end{aligned} \tag{4.56}$$

For plane stress, the coefficients D_{ij} are given by

$$
\begin{aligned}
D_{11} = D_{22} &= \frac{E}{1-\nu^2} \\
D_{12} = D_{21} &= \frac{\nu E}{1-\nu^2} \\
D_{33} = G &= \frac{E}{2(1+\nu)}
\end{aligned}
\tag{4.57}
$$

in which E, ν are modulus of elasticity and Poisson's ratio and E_m is the modulus related to the internal polar physics.

For elastic solids with isothermal assumptions, the energy equation is eliminated. We can also eliminate the entropy inequality from the mathematical model, thus (4.55) reduce to

$$
\begin{aligned}
\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} + \frac{\partial({}_a\sigma_{yx})}{\partial y} &= 0 \\
\frac{\partial({}_s\sigma_{xy})}{\partial y} + \frac{\partial({}_s\sigma_{yy})}{\partial y} - \frac{\partial({}_a\sigma_{yx})}{\partial x} &= 0 \\
\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) &= 0
\end{aligned}
\tag{4.58}
$$

The final mathematical model for the plane stress case consists of (4.58), (4.56), and (4.57). These are nine first order partial differential equations in nine dependent variables: u_1, u_2 ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, m_{xz}, m_{yz}, and Θ_z, hence the mathematical model has closure.

4.2.3 Dimensionless form of the mathematical model in $\mathbb{R}^2$ for plane stress

We nondimensionalize the mathematical model presented in $\mathbb{R}^2$ for the plane stress case ((4.58), (4.56), and (4.57)). We rewrite (4.58), (4.56) and (4.57) with a hat ($\hat{}$) on all quantities indicating that the quantities have their usual dimensions in terms of length ($\hat{L}$), force ($\hat{F}$) and time ($\hat{t}$). If we choose L_0, F_0 and t_0 as reference values of length, force, and time then the dimensionless length, force, and time (L, F and t) are defined as

$$
L = \frac{\hat{L}}{L_0}, \qquad F = \frac{\hat{F}}{F_0}, \qquad t = \frac{\hat{t}}{t_0}
$$

If we choose L_0, $E_0 = \tau_0$, $m_0 = \frac{\tau_0}{L_0}$, hence $F_0 = \tau_0 L_0^2$ then the dimensionless form or the mathematical model (4.58), (4.56) and (4.57) becomes

$$
\begin{gathered}
\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} + \frac{\partial({}_a\sigma_{yx})}{\partial y} = 0 \\
\frac{\partial({}_s\sigma_{xy})}{\partial x} + \frac{\partial({}_s\sigma_{yy})}{\partial y} - \frac{\partial({}_a\sigma_{yx})}{\partial x} = 0 \\
\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) = 0 \\
{}_s\sigma_{xx} = D_{11}\frac{\partial u_1}{\partial x} + D_{12}\frac{\partial u_2}{\partial y} \\
{}_s\sigma_{yy} = D_{21}\frac{\partial u_1}{\partial x} + D_{22}\frac{\partial u_2}{\partial y} \\
{}_s\sigma_{xy} = D_{33}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\
m_{xz} = \left(\frac{E_0}{m_0 L_0}\right) E_m \frac{\partial}{\partial x}\left(\Theta_z\right) \\
m_{yz} = \left(\frac{E_0}{m_0 L_0}\right) E_m \frac{\partial}{\partial y}\left(\Theta_z\right) \\
D_{11} = D_{22} = \frac{E}{1-\nu^2}; \quad D_{12} = D_{21} = \frac{\nu E}{1-\nu^2}; \quad D_{33} = G = \frac{E}{2(1+\nu)} \\
\Theta_z = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x}\right)
\end{gathered}
\tag{4.59}
$$

In (4.59) we have used $E_m = \frac{\hat{E}_m}{E_0}$ hence $\frac{E_0}{m_0 L_0}$ is in fact one, but it has been left in the constitutive theory for the moment tensor for the sake of clarity. Equations (4.59) are a system of nine first order linear coupled differential equations in nine dependent variables u_1, u_2 ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, m_{xz}, m_{yz}, and Θ_z.

4.2.4 Computational framework for solutions of the model problems: Least squares finite element method

Even though theoretical or analytical solutions of (4.59) for some special simplified model boundary value problems may be possible, in the present work we consider numerical solutions of (4.59) for the two model problems considered here using finite element formulations based on the residual functional in which hierarchical local approximations are considered in higher order global differentiability scalar product spaces. Details are well documented in many references [97–104], hence are not repeated here.

4.2.5 Computational framework for solutions of the model problems: Galerkin Method/Weak Form

If we consider the dimensionless form of the mathematical model in $\mathbb{R}^2$ (4.59) we can substitute the balance of angular momenta

$$
{}_a\sigma_{yx} = -\frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y}\right) \tag{4.60}
$$

and constitutive theories for ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, and ${}_s\sigma_{xy}$ into the balance of linear momenta. We can also substitute the expression for rotation Θ_z into the constitutive theories for m_{xz} and m_{yz}. This results in the following (assuming constant material properties):

$$\frac{\partial}{\partial x}\left(D_{11}\frac{\partial u_1}{\partial x}+D_{12}\frac{\partial u_2}{\partial y}\right)+D_{33}\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}\right)=0 \tag{4.61}$$

$$D_{33}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_{21}\frac{\partial u_1}{\partial x}+D_{22}\frac{\partial u_2}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}\right)=0 \tag{4.62}$$

$$m_{xz}=\frac{1}{2}\left(\frac{E_0}{m_0L_0}\right)E_m\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right) \tag{4.63}$$

$$m_{yz}=\frac{1}{2}\left(\frac{E_0}{m_0L_0}\right)E_m\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right) \tag{4.64}$$

(4.61)–(4.64) is a system of 4 second order PDEs in 4 variables: u_1, u_2, m_{xz}, and m_{yz}. Multiplying each equation in (4.61)–(4.64) by an appropriate test function and integrating over a typical element Ω_e results in:

$$\begin{gathered}
\int_{\Omega_e}\left[\frac{\partial}{\partial x}\left(D_{11}\frac{\partial u_1}{\partial x}+D_{12}\frac{\partial u_2}{\partial y}\right)+D_{33}\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}\right)\right]\delta u_1 dA=0\\
\int_{\Omega_e}\left[D_{33}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_{21}\frac{\partial u_1}{\partial x}+D_{22}\frac{\partial u_2}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}\right)\right]\delta u_2 dA=0\\
\int_{\Omega_e}\left[\frac{1}{E_m}m_{xz}-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\right]\delta m_{xz}dA=0\\
\int_{\Omega_e}\left[\frac{1}{E_m}m_{yz}-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\right]\delta m_{yz}dA=0
\end{gathered} \tag{4.65}$$

Equations (4.65) are equivalent to (4.61)–(4.64) by the fundamental lemma of calculus of variations if an appropriate space of test functions is chosen.

Applying integration by parts to each of the second order terms in (4.65) gives

$$\begin{gathered}
\int_{\Omega_e}\left[D_{11}\frac{\partial u_1}{\partial x}\frac{\partial \delta u_1}{\partial x}+D_{12}\frac{\partial u_2}{\partial y}\frac{\partial \delta u_1}{\partial x}+D_{33}\left(\frac{\partial u_1}{\partial y}\frac{\partial \delta u_1}{\partial y}+\frac{\partial u_2}{\partial x}\frac{\partial \delta u_1}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x}\frac{\partial \delta u_1}{\partial y}+\frac{\partial m_{yz}}{\partial y}\frac{\partial \delta u_1}{\partial y}\right)\right]dA\\
=\oint_{\Gamma^e}\delta u_1 F_x d\Gamma\\
\int_{\Omega_e}\left[D_{33}\left(\frac{\partial u_1}{\partial y}\frac{\partial \delta u_2}{\partial x}+\frac{\partial u_2}{\partial x}\frac{\partial \delta u_2}{\partial x}\right)+D_{12}\frac{\partial u_1}{\partial y}\frac{\partial \delta u_2}{\partial y}+D_{22}\frac{\partial u_2}{\partial y}\frac{\partial \delta u_2}{\partial y}+\frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x}\frac{\partial \delta u_2}{\partial x}+\frac{\partial m_{yz}}{\partial y}\frac{\partial \delta u_2}{\partial x}\right)\right]dA\\
=\oint_{\Gamma^e}\delta u_2 F_y d\Gamma\\
\int_{\Omega_e}\left[\frac{1}{E_m}m_{xz}\delta m_{xz}+\frac{1}{2}\frac{\partial \delta m_{xz}}{\partial x}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\right]dA=\oint_{\Gamma}\delta m_{xz}\left\{\frac{1}{2}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\hat{\boldsymbol{n}}_x\right\}d\Gamma\\
\int_{\Omega_e}\left[\frac{1}{E_m}m_{yz}\delta m_{yz}+\frac{1}{2}\frac{\partial \delta m_{yz}}{\partial y}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\right]dA=\oint_{\Gamma}\delta m_{yz}\left\{\frac{1}{2}\left(\frac{\partial u_1}{\partial y}-\frac{\partial u_2}{\partial x}\right)\hat{\boldsymbol{n}}_y\right\}d\Gamma
\end{gathered} \tag{4.66}$$

Table 4.1: Primary and secondary variables for 2nd order system

PV	SV
u_1	F_x
v_1	F_y
m_{xz}	$\Theta_z \hat{\boldsymbol{n}}_x$
m_{yz}	$\Theta_z \hat{\boldsymbol{n}}_y$

where

$$
\begin{aligned}
F_x &= \left\{ \left(D_{11} \frac{\partial u_1}{\partial x} + D_{12} \frac{\partial u_2}{\partial y} \right) \hat{\boldsymbol{n}}_x + \left(D_{33} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) - \frac{1}{2} \left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} \right) \right) \hat{\boldsymbol{n}}_y \right\} \\
F_y &= \left\{ \left(D_{33} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) + \frac{1}{2} \left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} \right) \right) \hat{\boldsymbol{n}}_x + \left(D_{12} \frac{\partial u_1}{\partial x} + D_{22} \frac{\partial u_2}{\partial y} \right) \hat{\boldsymbol{n}}_y \right\}
\end{aligned}
\tag{4.67}
$$

Equations (4.66) are symmetric in u_1, u_2, m_{xz}, m_{yz} and δu_1, δu_2, δm_{xz}, δm_{yz}. Therefore (4.61)–(4.64) are self-adjoint, and (4.66) will result in stable, convergent computations if approximation functions belong to $H^1(\Omega)$ and suitable boundary conditions are applied. Note that if piecewise bilinear approximations are used, there may be issues with locking due to the last two equations in (4.66), however these can be avoided by choosing a higher degree polynomial basis for approximation, or reduced integration may be used. In all studies performed in this dissertation higher degree approximations are used.

Table 4.1 shows the primary and secondary variables for (4.66). Conjugate pairs u_1, F_x and u_2, F_y appear in the same manner as classical plane elasticity, however the specific expressions for F_x and F_y appear different due to the asymmetry of the stress tensor. The final two primary variables are components of the moment tensor m_{xz} and m_{xy}. It may seem unusual to have force-like (or stress-like) primary variables, but it is natural here due to the higher order nature of the governing differential equations.

4.2.5.1 Fourth-order system

Equations (4.61)–(4.64) can be rewritten as a system of two fourth-order equations by substituting the expressions for m_{xz} and m_{yz} from (4.63), (4.64) into (4.61), (4.62).

$$
\begin{aligned}
&\frac{\partial}{\partial x} \left(D_{11} \frac{\partial u}{\partial x} + D_{12} \frac{\partial v}{\partial y} \right) + D_{33} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
&\qquad - \frac{1}{4} \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial x} \left(E_m \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial y} \left(E_m \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) \right\} = 0 \\
&D_{33} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_{21} \frac{\partial u}{\partial x} + D_{22} \frac{\partial v}{\partial y} \right) \\
&\qquad + \frac{1}{4} \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} \left(E_m \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial y} \left(E_m \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right) \right\} = 0
\end{aligned}
\tag{4.68}
$$

By applying the fundamental lemma of calculus of variations we arrive at the following:

$$
\begin{aligned}
&\int_{\Omega^e} \delta u \left[\frac{\partial}{\partial x}\left(D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial v}{\partial y}\right) + D_{33}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right.\\
&\qquad \left. -\frac{1}{4}\frac{\partial}{\partial y}\left\{ \frac{\partial}{\partial x}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\right\}\right] dA = 0\\
&\int_{\Omega^e} \delta v \left[D_{33}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{21}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y}\right)\right.\\
&\qquad \left. +\frac{1}{4}\frac{\partial}{\partial x}\left\{ \frac{\partial}{\partial x}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\right\}\right] dA = 0
\end{aligned}
\tag{4.69}
$$

Applying integration by parts as we did previously yields

$$
\begin{aligned}
&\int_{\Omega_e} \left[D_{11}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + D_{12}\frac{\partial v}{\partial y}\frac{\partial \delta u}{\partial x} + D_{33}\left(\frac{\partial u}{\partial y}\frac{\partial \delta u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial \delta u}{\partial y}\right)\right.\\
&\qquad \left. -\frac{1}{4}\left(\frac{\partial}{\partial x}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\frac{\partial \delta u}{\partial y} + \frac{\partial}{\partial y}\left(E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\frac{\partial \delta u}{\partial y}\right)\right] dA = \oint_{\Gamma^e} \delta u F_x d\Gamma\\
&\int_{\Omega_e} \left[D_{33}\left(\frac{\partial u}{\partial y}\frac{\partial \delta v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial \delta v}{\partial x}\right) + D_{12}\frac{\partial u}{\partial y}\frac{\partial \delta v}{\partial y} + D_{22}\frac{\partial v}{\partial y}\frac{\partial \delta v}{\partial y}\right.\\
&\qquad \left. +\frac{1}{4}\left(\frac{\partial}{\partial x}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\frac{\partial \delta v}{\partial x} + \frac{\partial}{\partial y}\left(E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right)\frac{\partial \delta v}{\partial x}\right)\right] dA = \oint_{\Gamma^e} \delta v F_y d\Gamma
\end{aligned}
\tag{4.70}
$$

At this point, the terms which were originally second-order in u and v are now symmetric, however we must apply integration by parts again to the terms which are currently 3rd order in u and v and 1st order in δu and δv

$$
\begin{aligned}
&\int_{\Omega_e} \left[D_{11}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + D_{12}\frac{\partial v}{\partial y}\frac{\partial \delta u}{\partial x} + D_{33}\left(\frac{\partial u}{\partial y}\frac{\partial \delta u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial \delta u}{\partial y}\right)\right.\\
&\qquad \left. +\ \frac{1}{4}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial^2 \delta u}{\partial y \partial x} + E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial^2 \delta u}{\partial y^2}\right)\right] dA\\
&\qquad = \oint_{\Gamma^e} \delta u F_x d\Gamma + \oint_{\Gamma^e} \frac{\partial \delta u}{\partial y}\frac{1}{2} M_z d\Gamma\\
&\int_{\Omega_e} \left[D_{33}\left(\frac{\partial u}{\partial y}\frac{\partial \delta v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial \delta v}{\partial x}\right) + D_{12}\frac{\partial u}{\partial y}\frac{\partial \delta v}{\partial y} + D_{22}\frac{\partial v}{\partial y}\frac{\partial \delta v}{\partial y}\right.\\
&\qquad \left. -\frac{1}{4}\left(E_m \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial^2 \delta v}{\partial x^2} + E_m \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial^2 \delta v}{\partial x \partial y}\right)\right] dA\\
&\qquad = \oint_{\Gamma^e} \delta v F_y d\Gamma + \oint_{\Gamma^e} -\frac{\partial \delta v}{\partial x}\frac{1}{2} M_z d\Gamma
\end{aligned}
\tag{4.71}
$$

Primary Variable	Secondary Variable
u	F_x
v	F_x
$\frac{\partial u}{\partial y}$	$M_z/2$
-$\frac{\partial v}{\partial x}$	$M_z/2$

Where:

$$
\begin{aligned}
F_x &= \left\{ \left(D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial v}{\partial y} \right) \hat{\boldsymbol{n}}_x + \left(D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} \right) \right) \hat{\boldsymbol{n}}_y \right\} \\
F_y &= \left\{ \left(D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{1}{2}\left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} \right) \right) \hat{\boldsymbol{n}}_x + \left(D_{12}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} \right) \hat{\boldsymbol{n}}_y \right\}
\end{aligned}
\tag{4.72}
$$

Equations (4.71) are symmetric functionals in u, v and δu, δv and therefore will result in stable, convergent computations if approximation functions are chosen from $H^2(\Omega)$, i.e. functions which are globally continuous and differentiable up to order 1 in x and y. Details for how to construct finite element approximations of class C^{11} can be found in [107].

Remarks

Each of the computational frameworks presented have some advantages and drawbacks, and we remark on some of the key differences here.

1. One of the main strengths of the least squares finite element method is that it is constructed through the residual functional I which is a measure of how well the computed solution satisfies the governing differential equations. If the approximation space is chosen such that the integrals are Riemann, then as $I \to 0$, the computed solution approaches the theoretical solution. This applies to any well-posed differential operator, including self-adjoint, non-self-adjoint, and non-linear differential operators.

2. Because the least-squares functional is always quadratic regardless of the underlying differential equations, the resulting integral forms are symmetric and lead to unconditionally stable computations.

3. The computational framework based on the Galerkin method/weak form approach results in integral forms which are symmetric only for self-adjoint differential operators. Since the differential operators for internal polar solid continua used here are self-adjoint, the resulting integral forms are symmetric and the computational process is unconditionally stable.

4. The Galerkin method/weak form approach allows for reduced requirements on global differentiability due to the application of integration by parts.

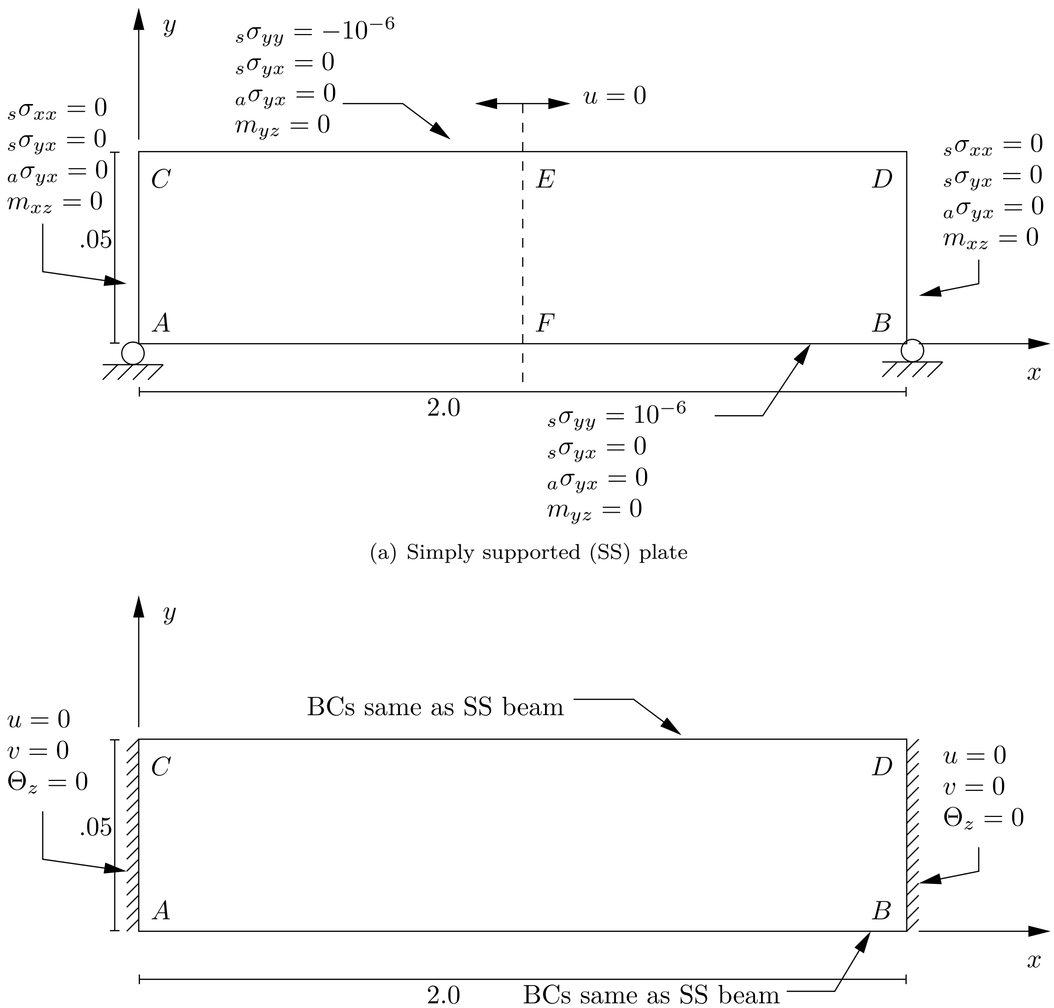
5. The first order system used in this study is comprised of 9 equations and 9 unknowns, while the second order system contains 4 equations and 4 unknowns and the fourth-order system

is 2 equations and 2 unknowns. The least squares process permits approximations of class C^{00} for the first order system, C^{11} for the second order system, and C^{33} for the fourth order system, while the Galerkin method/weak form approach only requires C^{00} for the second order system and C^{11} for the fourth order system. Therefore, the Galerkin method approach will generally require fewer computational resources if converged results can be obtained using lower p-levels, since higher order continuity approximation functions require higher p-levels (p=3 in the case of C^{11}, and p=7 in the case of C^{33}).

4.2.6 Model problems

We consider a thin plate with length $\hat{l}$ of 20 inches, width $\hat{b}$ of 0.5 inches and thickness $\hat{t}$ of 0.1 inches. With $L_0 = 10$ inches the dimensionless plate is $2 \times 0.05 \times 0.01$. We consider loads applied in the plane of the plate. We choose $\hat{E} = E_0 = 30 \times 10^6$ psi, hence $E = 1$. Dimensionless $E_m = \frac{\hat{E}_m}{E_0}$ is increased starting with 0.0 and is chosen to be a fraction of the dimensionless modulus of elasticity (which is unity). Clearly for $E_m = 0$, the internal polar physics is absent i.e. the usual small strain approximation theory of elasticity applies for this case.

Model problem 1 In this case we consider the plate to be simply supported as shown in figure 4.48 (a). Points A and B are constrained in the y direction, but are free to move in the x direction. On Face AB of the plate $\sigma_{yy} = 10^{-6}$ and on face CD, σ_{yy} of -10^{-6} is applied causing deflection of the plate in the negative y direction. At the center plane (EF) the x displacement is constrained (due to symmetry). Since $\hat{b}$ and $\hat{t}$ are much smaller than $\hat{l}$, the deformation behavior is similar to that of a simply supported slender beam (shear deformation is not significant). The domain ($l \times b$) 2×0.05 is modeled using a twenty element uniform discretization (ten elements along the length l and two elements along the width b) using nine node $p-$version hierarchical plane stress elements with higher order global differentiability local approximations in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces. Boundary conditions on the four boundaries of the domain $ABCD$ of the plate are also shown in figure $1(a)$. The nine node elements are mapped in a two unit square with the origin of the coordinate system ξ, η (natural coordinate system) at the center of the element. The element local approximation as well as all computations are performed using the natural coordinate system ξ, η. The degrees of local approximation in ξ and η (p_ξ, p_η) are chosen to be equal $p = p_\xi = p_\eta$ and are chosen to be the same for all dependent variables. Since the mathematical model is a system of first order partial differential equations, if the order of approximation space in x and y is chosen to be two i.e. local approximations of class C^1 in both x and y then the integrals over the discretization are Riemann. On the other hand, if we choose the order of the approximation space to be one, then the local approximations are of class C^0 implying that the integrals over the discretization are in Lebesgue sense. Due to the smoothness of the solution of the model problem, both choices work well, i.e. the C^0 solutions approach C^1 solutions upon convergence, but in the weak sense. In the results presented here we choose $k = 1$ i.e. local approximations of class C^0. A $p-$convergence study with $p = p_\xi = p_\eta = 3, 5, \ldots$ shows that at $p = 9$ the integrated sum of squares of the residuals are of the order of $O(10^{-16})$, confirming that the equations in the mathematical model are satisfied accurately in the pointwise sense. This is confirmed by the similar studies with solutions of class C^1 and their comparison with C^0 studies. Thus, we present results for $p = p_\xi = p_\eta = 9$ with local



(b) Clamped-clamped (CC) plate

Figure 4.48: Schematics and boundary conditions for simply supported and clamped-clamped plates (dimensionless)

approximations of class C^0 for all dependent variables using the 20 element uniform discretization described earlier.

Model problem 2 This model problem consists of the same plate as used in model problem 1 but is considered clamped at the two ends ($x = 0$ and $x = 2$ shown in figure 4.48 (b)). The boundary conditions on the boundaries AB and CD (excluding points A and B) remain the same as in model problem 1. Boundary conditions on AC and BD (clamped boundaries) are $u = v = \theta_z = 0$ as shown in figure 4.48 (b). The details of the discretization, choice of $p-$levels, choice of order of approximation space etc. are the same for this model problem as those described for model problem 1. In this case also, a $p-$convergence study for solutions of class C^0 yields integrated sum of squares of the residual of the order of $O(10^{-16})$ as in the case of model problem 1. Thus, for this model problem also, $p = p_\xi = p_\eta = 9$ and C^0 local approximations for all dependent variables yields very accurate solutions, hence are used to compute the results presented here.

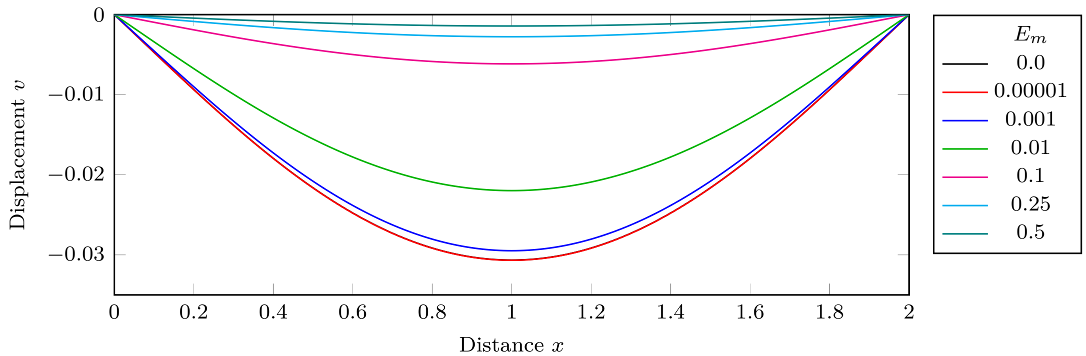


Figure 4.49: Displacement v versus distance x at $y = 0.025$ (simply supported plate)

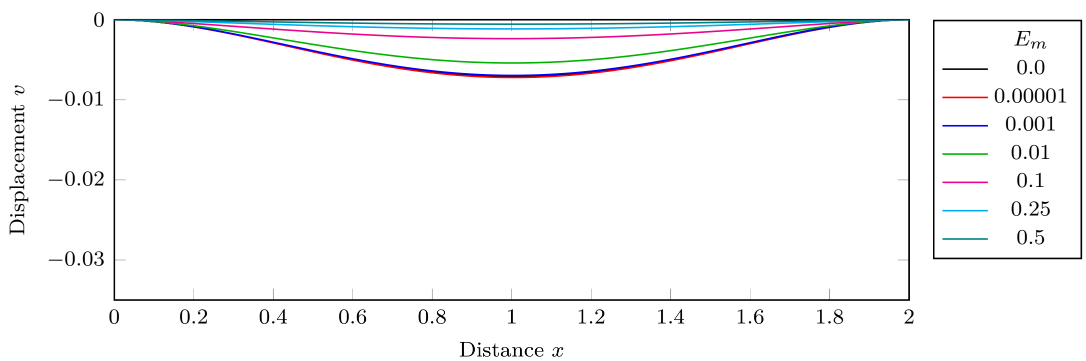


Figure 4.50: Displacement v versus distance x at $y = 0.025$ (clamped-clamped plate)

Solution for model problems 1 and 2   In the solutions presented here for model problems 1 and 2 the dimensionless modulus of elasticity of 1 corresponds to $\hat{E} = 30 \times 10^6$ psi. When E_m, the dimensionless material coefficient related to the internal polar physics, is zero the internal polar physics is completely absent and we have standard equations in the mathematical model for plane stress behavior based on infinitesimal theory of elasticity. We choose Poisson's ratio $\nu = 0.3$. Values of E_m used here range from 0.00001–0.5. Progressively increasing values of E_m reflect progressively

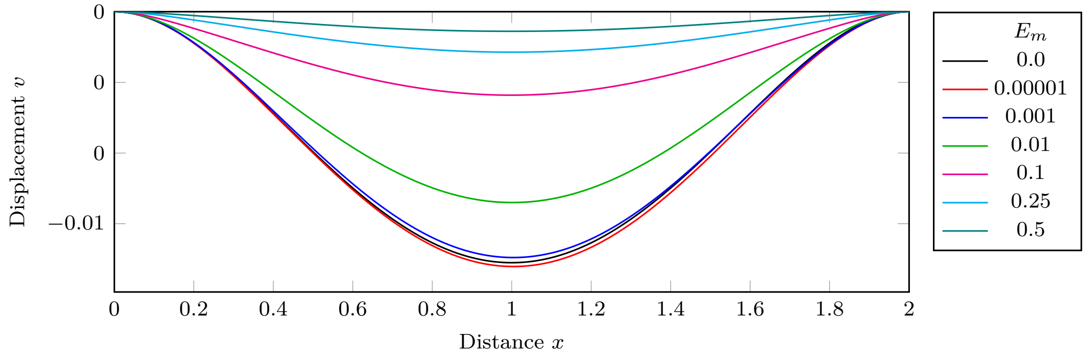


Figure 4.51: Displacement v versus distance x at $y = 0.025$ (clamped-clamped plate)

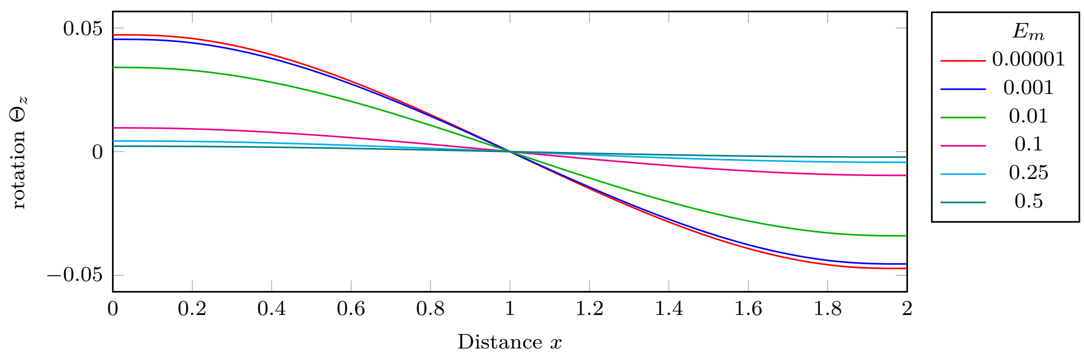


Figure 4.52: Rotation Θ_z versus distance x at $y = 0.025$ (simply supported plate)

increasing influence of internal polar physics. Figure 4.49 shows plots of displacement v versus x at the centerline ($y = 0.025$) for the simply supported (SS) plate of (figure 4.48 (a)). For $E_m = 0$, the solution is in agreement with Timoshenko beam theory. $E_m = 0.00001$, representing extremely small influence of internal polar physics, hardly has any influence on the deflection (as expected). For $E_m = 0.001, 0.01, 0.1, 0.25$, and 0.5 we observe progressively reducing vertical displacement of v of the plate centerline due to progressively increased resistance to deformation due to progressively increased influence of internal polar physics. The displacements v of the bottom and the top faces of the plate ($y = 0.0$ and $y = 0.05$) are virtually the same as the displacement v of the centerline ($y = 0.025$) of the plate as expected for a slender beam like what is used here. Graphs of v versus x at the centerline of the clamped (CC) plate of figure 4.48 (b) for the same σ_{yy} and the same values of E_m as used for the SS plate are shown in figure 4.50. These are plotted using the same x, y scales as in figure 4.49. Substantially reduced displacement v values for all values of E_m compared to the SS plate are obvious. For $E_m = 0$, the deflection v is in agreement with Timoshenko beam theory. The purpose of the results in figure 4.50 is to compare directly with the results in figure 4.49 for the

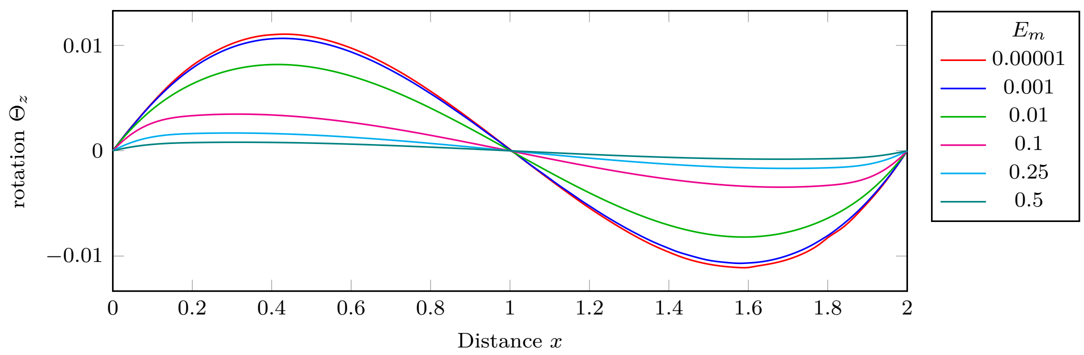


Figure 4.53: Rotation Θ_z versus distance x at $y = 0.025$ (clamped-clamped plate)

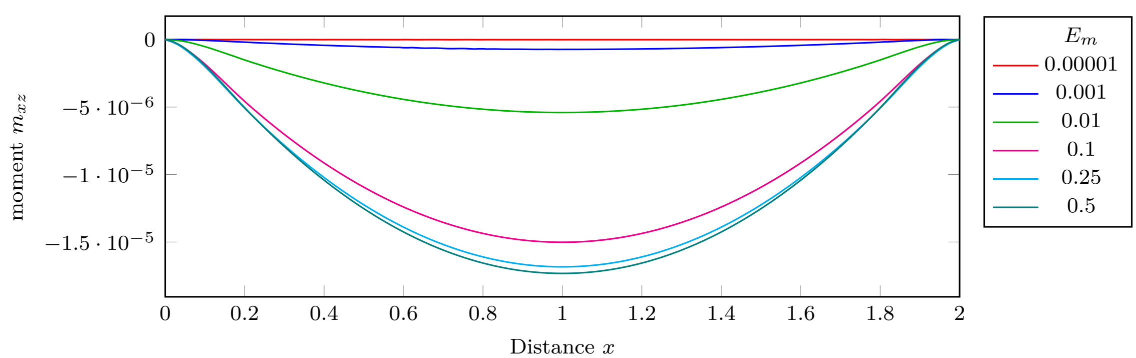


Figure 4.54: Moment m_{xz} versus distance x at $y = 0.025$ (simply supported plate)

SS case so that the substantial reduction in v for the CC plate can be observed easily. Figure 4.51 shows the same results as in figure 4.50 but using an enlarged scale for the y axis for more clarity. Behavior is similar to the SS plate i.e. progressively increasing values of E_m result in progressively reduced displacement v due to progressively increasing resistance to deformation offered by the internal polar physics. In figures 4.49 and 4.50 the results are symmetric about $x = 1.0$ due to symmetry of geometry, loading, and boundary conditions.

Plots of rotation Θ_z versus x at $y = 0.025$ for SS and CC plates for $E_m = 0.00001, 0.001, 0.01, \ldots, 0.5$ are shown in figures 4.52 and 4.53. We make some observations and remarks.

(a) When $E_m = 0.00001$, the internal polar physics is virtually absent, hence Θ_z and its gradient in the $x-$direction have the largest magnitude for the SS plate as well as the CC plate compared to the higher values of E_m as for $E_m = 0.00001$ the internal polar resistance to deformation is minimal.

(b) As E_m increases Θ_z reduces due to progressively increasing resistance offered by the progressively increasing influence of internal polar physics.

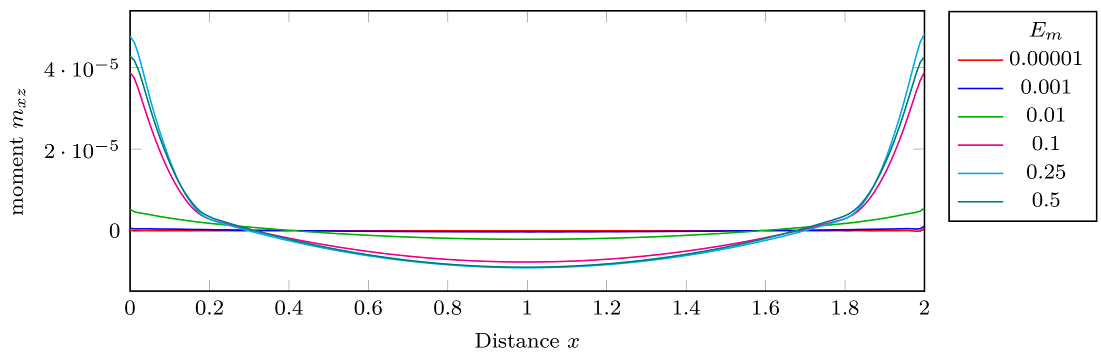


Figure 4.55: Moment m_{xz} versus distance x at $y = 0.025$ (clamped-clamped plate)

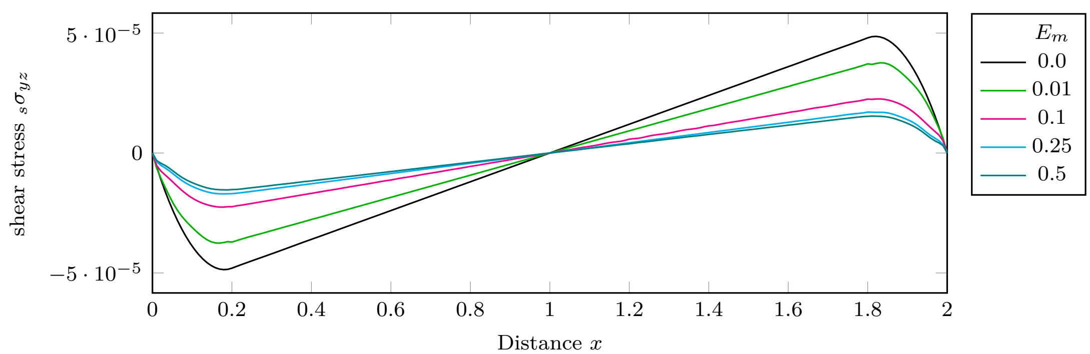


Figure 4.56: Shear stress ${}_s\sigma_{yx}$ versus distance x at $y = 0.025$ (simply supported plate)

(c) Even though Θ_z and its gradients are the highest for $E_m = 0.00001$, the internal resistance due to internal polar physics is smallest for this value of E_m compared to all other higher values used here.

(d) Antisymmetry of Θ_z about $x = 1.0$ (as expected) is quite obvious from the graphs.

Figures 4.54 and 4.55 show plots of moment m_{xz} versus x at $y = 0.025$ for SS and CC plate for the same values of E_m used in figures 4.52 and 4.53. We observe that

(i) For $E_m = 0.00001$, m_{xz} has the lowest value for both SS and CC plates even though Θ_z and $\Theta_{z,x}$ have the largest values (figures 4.52 and 4.53). This is of course due to the fact that such a low value of E_m implies virtually no internal polar physics, hence virtually no resistance to rotations, thus resulting in extremely small values of moment m_{xz}.

(ii) As E_m increases, Θ_z reduces but m_{xz} increases due to increased contribution of internal polar physics, hence progressively increasing resistance to rotations.

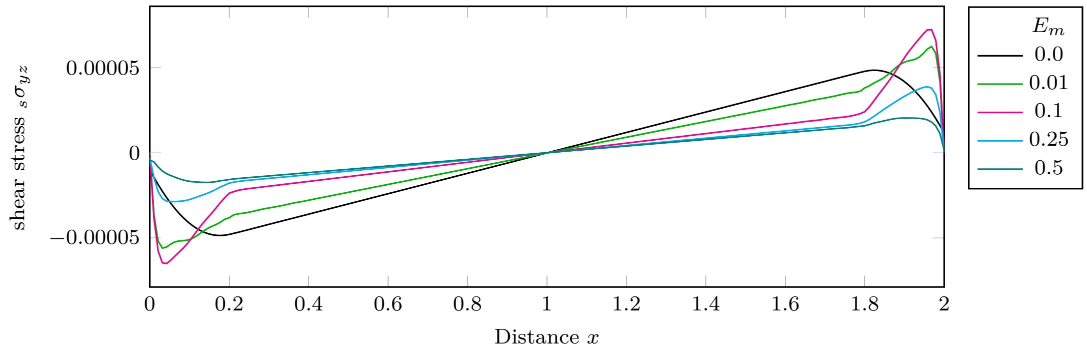


Figure 4.57: Shear stress $_s\sigma_{yx}$ versus distance x at $y = 0.025$ (clamped-clamped plate)

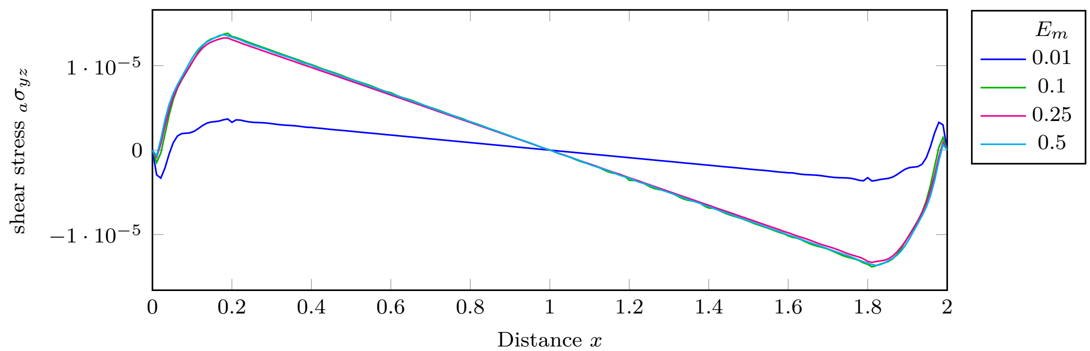


Figure 4.58: Shear stress $_a\sigma_{yx}$ versus distance x at $y = 0.025$ (simply supported plate)

(iii) We note that for $E_m = 0.5$, Θ_z and $\Theta_{z,x}$ are the lowest (figures 4.52 and 4.53) but the corresponding m_{xz} (figures 4.54 and 4.55) have the highest values due to increased resistance to deformation offered by the pronounced influence of internal polar physics.

(iv) Symmetry of m_{xz} about $x = 1.0$ is clearly observed.

(v) In the absence of internal polar physics m_xz would be zero as evidenced by m_{xz} values for $E_m = 0.00001$ for which Θ_z and its gradient in the x direction are the largest.

(vi) The existence of the extent of internal polar physics is dependent on the constitution of the matter. In the simplified constitutive theory used in the model problems the material coefficient E_m is the measure of the extent of internal polar physics.

Graphs of $_s\sigma_{yx}$ in figures 4.56 and 4.57 for SS and CC and those of $_a\sigma_{yx}$ in figures 4.58 and 4.59 for SS and CC plates at the centerline are shown for progressively increasing values of E_m: $E_m = 0.0, 0.01, 0.1, 0.25,$ and 0.5 for figures 4.56 and 4.57 and $E_m = 0.01, 0.1, 0.25,$ and 0.5 for figures 4.58 and 4.59. Both $_s\sigma_{yx}$ and $_a\sigma_{yx}$ are antisymmetric about $x = 1.0$ as expected. Since internal polar

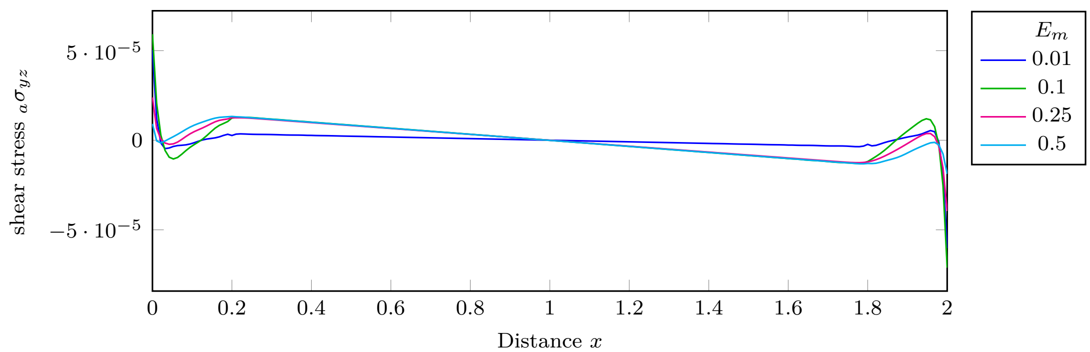


Figure 4.59: Shear stress ${}_a\sigma_{yx}$ versus distance x at $y = 0.025$ (clamped-clamped plate)

physics influences displacements and their gradients, ${}_s\sigma_{yx}$ purely due to non-polar physics when $E_m = 0$ is influenced by the presence of internal polar physics as evident in figures 4.56 and 4.57. Of course, in the absence of internal polar physics, ${}_a\sigma_{yx}$ and m_{xz} would be zero. With progressively increasing influence of internal polar physics (progressively increasing values of E_m) ${}_a\sigma_{yx}$ values along the length of the plate increase in magnitude as expected (just like the moment m_{xz}).

5. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

5.1 Summary and conclusions

The polar theory presented here for isotropic, homogeneous solid and fluent continua is motivated by the fact that the polar decomposition of changing displacement gradient tensor and velocity gradient tensor at a location and its neighbors can result in different rotations and rates of rotations at neighboring locations which, if resisted by the continua, result in conjugate internal moments. These internal rotations and rates of rotations and the conjugate internal moments can result in additional energy storage and energy dissipation. The currently used thermodynamic framework for isotropic, homogeneous solid and fluent continua completely ignores this physics in the derivation of the conservation and balance laws. The theory resulting from the new physics considered here is in fact 'a polar theory' as it considers rotations or rates of rotations and moments as a conjugate pair. The rates of rotations are internal and are completely defined using skew-symmetric part of the velocity gradient tensor, and the rotations are completely defined using the skew-symmetric part of the displacement gradient tensor, thus this theory does not require rotations as external degrees of freedom. The thermodynamic framework resulting from this new theory is obviously a more complete thermodynamic framework for isotropic, homogeneous fluent continua as it incorporates additional physics due to internal rates of rotations in the derivation of conservation and balance laws that is completely ignored in the presently used framework. In fact, the currently used thermodynamic framework is a subset of the more complete thermodynamic framework presented in this dissertation resulting from the polar theory.

Derivation of conservation and balance laws have been presented for polar fluent continua in contravariant and covariant bases and in Jaumann rates using Cauchy stress tensor, Cauchy moment tensor, heat vector, Helmholtz free energy density, and entropy density. The derivations exhibit the following features: (i) Cauchy stress tensor is nonsymmetric (ii) Cauchy moment tensor is symmetric due to the moment of moments (or couples) balance law (iii) Decomposition of Cauchy stress tensor into symmetric and antisymmetric tensors shows that (a) symmetric Cauchy stress tensor and symmetric part of the velocity gradient tensor are conjugate (due to energy equation and entropy inequality) (b) antisymmetric part of the Cauchy stress tensor is balanced by the gradients of the Cauchy moment tensor (due to balance of angular momenta). (iv) Cauchy moment tensor and symmetric part of the gradient of rate of rotation tensor are conjugate (due to energy equation and entropy inequality) (v) It is shown that the constitutive theories for symmetric Cauchy stress tensor, Cauchy moment tensor, heat vector and the thermodynamic relations for specific internal energy and others provide closure to the mathematical model presented here. Rate constitutive theories for the Cauchy stress tensor, Cauchy moment tensor, and heat vector are derived for compressible and incompressible polar thermofluids. Derivation of rate constitutive theories of up to order n is presented for the symmetric part of the deviatoric Cauchy stress tensor using the theory of invariants and generators. A rate constitutive theory of order one is presented for the Cauchy moment tensor. Simplified forms of the constitutive theories for the cases where the stress and moment tensor are quadratic functions of their argument tensors, as well as linear functions of their argument tensors are also presented.

Derivations of conservation and balance laws are also presented for polar solid continua in Lagrangian description for small deformation. The following observations can be made from the derivations: (i) Cauchy stress tensor is nonsymmetric (ii) Cauchy moment tensor is symmetric due to the moment of moments (or couples) balance law (iii) Decomposition of the Cauchy stress tensor into symmetric and antisymmetric tensors shows that (a) symmetric Cauchy stress tensor and infinitesimal Green's strain tensor are conjugate (due to energy equation and entropy inequality) (b) antisymmetric part of the Cauchy stress tensor is balanced by the gradients of the Cauchy moment tensor (due to balance of angular momenta). (iv) The Cauchy moment tensor and symmetric part of the gradient rotation tensor are conjugate (due to energy equation and entropy inequality) (v) It is shown that the constitutive theories for symmetric Cauchy stress tensor, Cauchy moment tensor, heat vector and the thermodynamic relations for specific internal energy and others provide closure to the mathematical model presented here. Constitutive theories for the Cauchy stress tensor, Cauchy moment tensor, and heat vector are presented using two approaches. Approach I is based on the energy equation and entropy inequality in their original forms, while Approach II considers strain energy density separately as it does not contribute to rate of entropy production. Alternate derivations are presented for Approach I using Helmholtz free energy density and conditions resulting from the entropy inequality as well as the theory of invariants and generators. For approach II derivations are presented using (i) theory of generators and invariants, (ii) strain energy density, (iii) complementary strain energy and (iv) Taylor series expansion.

We emphasize that the polar theories presented here are not micropolar theories (as mentioned in section 1). The theories presented here are for isotropic, homogeneous solid and fluent continua in which varying rotations, rates of rotations, and their gradients can result in additional energy storage and energy dissipation. The polar theory presented in this paper is inherently local and hence not capable of capturing nonlocal effects. We remark that the polar continuum theory presented in this dissertation is not to be labeled as a "stress couple theory" (see remarks in section 2.2.3). Rate of dissipation due to rates of rotations necessitates existence of conjugate moment tensor. It is only after the balance of angular momenta we realize that only the antisymmetric part of the Cauchy stress tensor is balanced by the gradients of the Cauchy moment tensor. We note that the existence of the Cauchy moment tensor is established long before we realize a relationship between its gradients and the antisymmetric part of the Cauchy stress tensor. The polar continuum theories presented here based on rotations and rates of rotation gradients are not the same as the strain gradient and strain rate gradient theory (see sections 3.1.2 and 2.1.3). The internal polar continuum theories for solid and fluent continua are an extension to classical continuum mechanics

5.2 Limitations, recommendations and future work

There are a number of potential future studies that can be carried out to extend the theories presented here. The most obvious need is for experimental work to determine material coefficients for the constitutive theories for the moment tensor. In the simplest case of a linear constitutive theory, ${}^m\mu$ and ${}^m\lambda$ must be determined for polar solids; and ${}^m\eta$ and ${}^m\kappa$ must be determined for polar thermofluids. Note that ${}^m\kappa$ must be determined even in the case of incompressible polar thermofluids as $\mathrm{tr}([{}^\Theta D])$ does not vanish in a divergence free velocity field. If the flow field only varies in two dimensions, ${}^m\kappa$ plays no role in deformation since the out of plane gradients are zero. In such experiments it would be beneficial to isolate the polar physics from the non-polar physics. For example, a polar solid subject to uniaxial tension would not show any effect from the polar physics because the rotation gradients are zero. Similarly, a polar fluid experiencing Couette flow would show no polar effect as long as the boundaries are incapable of constraining the rotation rate of the fluid particles. The numeric methods described in chapter 3 will be useful in designing experiments to find ${}^m\mu$ and ${}^m\eta$, and similar methods in $\mathbb{R}^3$ can be used to assist design of experiments to determine ${}^m\lambda$ and ${}^m\kappa$.

One of the key limitations of the theory presented here for internal polar solid continua is that the conservation and balance laws and constitutive theories derived are only valid for small deformations and small rotations. The rotation tensor gradient used in this work was derived from the displacement gradient tensor, and its components are gradients of rotations about the coordinate axes in the reference configuration. In order to be valid for finite deformation, an alternate derivation of the rotation gradient tensor is required. Possible approaches include a deriving a rotation measure similar to the way Green's strain is derived or considering rotations involving the covariant or contravariant base vectors. Such a theory should result in a measure of relative rotations that is basis dependent. A basis dependent rotation measure would of course lead to higher order convected time derivatives, which would permit higher order rate constitutive theories for the Cauchy moment tensor in both Lagrangian and Eulerian descriptions. This would lead to constitutive theories for internal polar thermoviscoelastic solids with and without memory, as well as internal polar thermoviscoelastic fluids with and without memory. Constitutive theories for polar thermoviscoelastic fluids similar to the Maxwell, Drolly-B, and Genius models could be derived. Constitutive theories for polar thermoviscoelastic solids similar to the Kelvin-Voigt model could also be derived. Note that the framework presented here permits higher order rate constitutive theories for polar thermofluids, but only for the Cauchy stress tensor and heat vector.

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